MATHEMATICS

N. S. BAKHVALOV

1957

SovietRxiv

Abstract

Full Text

MATHEMATICS

N. S. BAKHVALOV

ON THE QUESTION OF THE NUMBER OF ARITHMETIC OPERATIONS IN SOLVING THE POISSON EQUATION FOR A SQUARE BY THE METHOD OF FINITE DIFFERENCES

(Presented by Academician S. L. Sobolev on 10 X 1956)

For every problem of mathematical physics there exist various methods of approximate solution, requiring, in order to determine the solution with accuracy ε, one or another number of arithmetic operations, increasing as $\varepsilon \to 0$. The use of mathematical machines makes it possible to increase the number of operations performed and to find a solution with greater accuracy. Therefore the following problem becomes important: to investigate methods from the point of view of the growth of the number of arithmetic, respectively logical, operations as $\varepsilon \to 0$, and to find methods with the least possible growth in the number of operations.

Consider the system of linear equations

$$L_{ij}(v_{ij}) = \frac{v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1} - 4v_{ij}}{h^2} = f_{ij}$$

$$\text{for } 0 < i, j < N \quad (N \text{ an integer; } h = 1/N); \tag{1}$$

$$v_{ij} = \varphi_{ij} \quad \text{for } 0 \le i, j \le N, \text{ if } ij(N-i)(N-j) = 0,$$

obtained when solving by the method of finite differences the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

in the square $0 \le x, y \le 1$ with the Dirichlet boundary condition

$$u|_\Gamma = \varphi(x, y)|_\Gamma.$$

Here by $v_{ij}, \varphi_{ij}, \ldots$ we mean the values of the functions $v(x, y), \varphi(x, y), \ldots$ at the point (ih, jh).

In the paper (^{1}) it is shown that, in order to find the solution of system (1) by the method of successive over-relaxations, it is sufficient to perform $\sim N^3 |\log \varepsilon|$ additions (^{2}).

If the difference analogues (^{3}) of the Green functions for the Dirichlet and Poisson problems are known, then the direct writing out of the values of all v_{ij} $(0 < i, j < N)$ requires $\sim N^4$ additions, $\sim N^4$ multiplications, and the use of $\sim N^4$ auxiliary numbers. In what follows we shall omit the word "difference analogue."

Let $N = 2^s$. Then, if the Green functions for the Dirichlet and Poisson problems for the squares $0 \le i, j \le 2^r$ $(r = 1, 2, \ldots, s)$ are known, one can compute the values of all v_{ij}, performing $\sim N^2 \log N$ additions and $\sim N^2 \log N$ multiplications. We shall show how to do this.

Let Z_{ij}^r $(r = 0, 1, \ldots, s)$ be functions that vanish when $\left\{\frac{i}{2^r}\right\}\left\{\frac{j}{2^r}\right\} = 0$ and satisfy the equations

$$L_{ij}(Z_{ij}^r) = f_{ij}$$

for the remaining i and j $(0 < i, j < N)$.

Set $Z_{ij}^r - Z_{ij}^{r-1} = \omega_{ij}^r$ $(r = 1, 2, \ldots, s)$. Determining the values $L_{ij}(\omega_{ij}^{r+1})$ for various i and j such that $\left\{\frac{i}{2^{r+1}}\right\}\left\{\frac{j}{2^{r+1}}\right\} \ne 0$, we have

$$L_{ij}(\omega_{ij}^{r+1}) = \begin{cases} 0, & \text{if } \left\{\frac{i}{2^r}\right\}\left\{\frac{j}{2^r}\right\} \ne 0; \\ f_{ij} - \dfrac{Z_{i+1,j}^r + Z_{i-1,j}^r}{h^2}, & \text{if } \left\{\frac{i}{2^r}\right\} = 0; \\ f_{ij} - \dfrac{Z_{i,j+1}^r + Z_{i,j-1}^r}{h^2}, & \text{if } \left\{\frac{j}{2^r}\right\} = 0. \end{cases} \tag{2}$$

It is obvious that

$$\omega_{ij}^1 = \begin{cases} -f_{ij}\dfrac{h^2}{4}, & \text{if } \left\{\frac{i}{2}\right\}\left\{\frac{j}{2}\right\} \ne 0; \\ 0, & \text{if } \left\{\frac{i}{2}\right\}\left\{\frac{j}{2}\right\} = 0. \end{cases}$$

Suppose that ω_{ij}^l $(l = 1, 2, \ldots, r)$ are known for $\left\{\frac{i}{2^l}\right\}$ or $\left\{\frac{j}{2^l}\right\}$ equal to $1/2$. Using the Green functions of the Dirichlet problem for the squares $0 \le i, j \le 2^{l-1}$ $(l = 1, 2, \ldots, r)$, we find the values $Z_{ij}^r = \sum_{l=1}^r \omega_{ij}^l$ needed for determining

$L_{ij}(Z_{ij}^{r+1})$ by formula (2). Then, with the aid of the Green function of the Poisson problem for the square $0 \le i, j \le 2^{r+1}$, we obtain the values ω_{ij}^{r+1} for $\left\{\frac{i}{2^{r+1}}\right\}$ or $\left\{\frac{j}{2^{r+1}}\right\}$ equal to 1/2. In this way we find the values ω_{ij}^{r} for $\left\{\frac{i}{2^{r}}\right\}$ or $\left\{\frac{j}{2^{r}}\right\}$ equal to 1/2, for $r = 2, 3, \dots, s$.

Let w_{ij} be the solution of system (1) for $f_{ij} \equiv 0$. Put

$$p_{ij}^{r} = w_{ij} + \sum_{l=r}^{s} \omega_{ij}^{l} \qquad (r = 1, 2, \dots, s). \tag{3}$$

It is clear that $p_{ij}^{r} = v_{ij}$ if $\left\{\frac{i}{2^{r-1}}\right\}\left\{\frac{j}{2^{r-1}}\right\} = 0$, and $L_{ij}(p_{ij}^{r}) = 0$ for the remaining i and j.

Using the Green function of the Dirichlet problem for the square $0 \le i, j \le 2^{s}$, we determine $w_{2^{s-1},j}$ $(0 < j < N)$ and $w_{i,2^{s-1}}$ $(0 < i < N)$. From (3) we find $p_{2^{s-1},j}^{s} = v_{2^{s-1},j}$ $(0 < j < N)$ and $p_{i,2^{s-1}}^{s} = v_{i,2^{s-1}}$ $(0 < i < N)$. Now, with the aid of the Green function of the Dirichlet problem for the square $0 \le i, j \le 2^{s-1}$, we determine p_{ij}^{s}, and then $p_{ij}^{s-1} = v_{ij}$ for $\left\{\frac{i}{2^{s-1}}\right\}$ or $\left\{\frac{j}{2^{s-1}}\right\}$ equal to 1/2, etc. Thus we obtain the values v_{ij} at all points of the square $0 \le i, j \le N$.

It is easy to calculate, taking into account the symmetry of the Green function of the Poisson problem and the symmetry of the square with respect to its four axes of symmetry, that for the computations indicated above it is sufficient to know somewhat fewer than $2N^2$ auxiliary numbers—the values of the Green functions of the Dirichlet and Poisson problems for the squares $0 \le i, j \le 2^{r}$ $(r = 1, 2, \dots, s)$.

Obviously,

$$|\omega_{ij}^{r}| \le \overline{\omega}_{ij}^{r},$$

where $\bar{\omega}_{ij}^{r}$ are functions obtained analogously to ω_{ij}^{r} when f_{ij} is replaced by $-\max|f_{ij}|$ $(0 < i, j < N)$. Hence it follows that

$$\sum_{r=1}^{s} |\omega_{ij}^{r}| \leqslant \sum_{r=1}^{s} \bar{\omega}_{ij}^{r} = \Omega_{ij}^{s},$$

where Ω_{ij}^{s} is the solution of system (1) for $\varphi_{ij} = 0$ and with f_{ij} replaced by $-\max|f_{ij}|$ $(0 < i, j < N)$.

The functions Ω_{ij}^{s} are uniformly bounded as s varies, and therefore, in computing v_{ij} by the method indicated above, no loss of significant digits occurs.

In the case of domains of arbitrary shape, the method presented makes it possible, by performing $\sim N^2 \log N$ additions and $\sim N^2 \log N$ multiplications, to reduce the problem of solving the system of finite-difference equations corresponding to the Poisson equation to the solution of the system of equations corresponding to the Laplace equation.

Moscow State University
named after M. V. Lomonosov

Received
5 X 1956

REFERENCES

1. D. Young, *Trans. Am. Math. Soc.*, **76**, No. 1 (1954).
2. A. N. Kolmogorov, *DAN*, **108**, No. 3 (1956).
3. R. Courant, K. Friedrichs, H. Lewy, *Uspekhi Mat. Nauk*, vol. 8, 125 (1940).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.