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Abstract

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MATHEMATICS

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ASYMPTOTICS OF THE SOLUTION OF A SECOND-ORDER PARABOLIC PARTIAL DIFFERENTIAL EQUATION WITH A SMALL PARAMETER AT THE HIGHEST DERIVATIVE

(Presented by Academician S. L. Sobolev on 21 VI 1957)

In the present note we consider the behavior, as $\varepsilon \to 0$ ($\varepsilon > 0$), of the solution of the following problem (Cauchy problem):

$$L_\varepsilon u^\varepsilon(x,t) \equiv \varepsilon u^\varepsilon_{xx} - u^\varepsilon_t + b(x,t)u^\varepsilon_x + c(x,t)u^\varepsilon = 0; \tag{1}$$

$$u^\varepsilon(x,t)\big|_{t=0} = \psi(x), \qquad (x,t) \in D_\infty\ (-\infty < x < \infty,\ 0 \le t \le T), \tag{2}$$

where:

a) $\psi^k(x)$ are continuous for $x \neq 0$; there exists

$$\lim_{x\to\pm 0} \psi^{(k)}(x) = \psi^{(k)}(\pm 0),$$

$\psi(+0) \neq \psi(-0)$, $k = 0, 1, \dots, n,\ n \ge 1$;
$|\psi^{(n)}(x) - \psi^{(n)}(-0)| < C|x|$ for $-\delta \le x \le 0$;
$|\psi^{(n)}(x) - \psi^{(n)}(+0)| < Cx$ for $0 \le x \le \delta$, where $C > 0$, $\delta > 0$ is arbitrarily small;

b) $\psi^{(r)}(x)$, $r = 0, 1, \dots, n$, as $|x| \to \infty$ grow no faster than $\exp(C_0 x^2)$, $C_0 > 0$ being some constant (for simplicity, all subsequent arguments are carried out under the assumption that $\psi^{(r)}(x)$ are bounded);

c) the functions $b(x,t)$, $b_x(x,t)$, $b_t(x,t)$, $b_{xx}(x,t)$, $c(x,t)$ in D_∞ satisfy a Lipschitz condition in x and t; moreover, $b(x,t)$, $b_x(x,t)$, $c(x,t)$ are assumed bounded.

Denote by $v(x,t)$ the solution of (1), (2) for $\varepsilon = 0$:

$$L_0 v(x,t) \equiv -v_t + bv_x + cv = 0; \tag{1°}$$

$$v(x,t)\big|_{t=0} = \psi(x), \qquad (x,t) \in D_\infty, \tag{2°}$$

and by $l(x,t)$ the characteristic of equation (1°) passing through the point (x,t).

In the degeneration of differential equations of higher order into equations of lower order, on the one hand, there is a loss of part of the boundary conditions, which gives rise to the phenomenon of a "boundary layer" ; on the other hand, there may occur a loss of smoothness for the solution of the degenerate equation. In the first case, near that part of the boundary where the loss of conditions for the solution of the nondegenerate equation occurs, one can write an asymptotic expansion (see, for example, (1)). The second case occurs in the problem considered here. In this case, in a neighborhood of the characteristic $l(0,0)$, on which the loss of smoothness of the solution $v(x,t)$ of problem (1°), (2°) occurs, one may observe, for the solution $u^\varepsilon(x,t)$, the phenomenon of an "internal boundary layer." In this paper the asymptotics of the solution $u^\varepsilon(x,t)$ near $l(0,0)$ is given.

Definition. The part principal with respect to ε, as $\varepsilon \to 0$, of $u^\varepsilon(x,t) - v(x,t)$ inside some neighborhood of $l(0,0)$ will be called the **internal parabolic boundary layer**.

Passing from x and t to new variables y and z, where z is the length of the arc of the characteristic $l(x,t)$ between the points $(y,0)$ and (x,t), and y is the distance along the x-axis from the origin to the point of intersection of $l(x,t)$ with $t=0$, and retaining the previous notation, we write (1), (2) in the form

$$L_\varepsilon u^\varepsilon(z,y) = \varepsilon a(z,y) u^\varepsilon_{yy} - u^\varepsilon_z + c'(z,y) u^\varepsilon; \tag{1'}$$

$$u^\varepsilon(z,y)\big|_{z=0} = \psi(y), \qquad -\infty < y < \infty, \qquad 0 \le z \le Z, \tag{2'}$$

where $a(z,y) = (1+b^2)^{-1/2}$, $c'(z,y) = c(1+b^2)^{-1/2}$. For what follows it is sufficient to consider the case when $\psi(x) \equiv 0$ for $x > 0$. Let now $h^+(x) = 1$ for $x \ge 0$ and $h^+(x) = 0$ for $x < 0$; $h^-(x) = 1 - h^+(x)$, D_R is $(|x| \le R,\ 0 \le t \le T)$; D'_R is $(|y| \le R,\ 0 \le z \le Z)$.

Lemma. If b and c are constants and conditions a), b) are satisfied, then for the solution u^ε of problem (1), (2) in a neighborhood of $l(0,0)$ (in the variables y, z, in a neighborhood of $y=0$) the following representation holds:

$$u^\varepsilon(z,y) = v(z,y) + v_{0\varepsilon}(z,y) + \sum_{k=1}^{n} \left(v_k^+(z,y,\varepsilon) + v_k^-(z,y,\varepsilon)\right) + O\left(\sqrt{\varepsilon}^{\,n+1}\right), \quad (3)$$

where $v(z,y)$ is the solution of $(1')$, $(2')$ for $\varepsilon = 0$;

$$v_{0\varepsilon}(z,y) = \frac{\psi(-0)}{\sqrt{\pi}}\, h^+(y) \int_{y/2\sqrt{\varepsilon z}}^{\infty} e^{-\xi^2}\, d\xi - \frac{\psi(-0)}{\sqrt{\pi}}\, h^-(y) \int_{-\infty}^{y/2\sqrt{\varepsilon z}} e^{-\xi^2}\, d\xi$$

is the boundary layer defined above;

$$v_k^+(z,y,\varepsilon) = \frac{\psi^{(k)}(-0)}{\sqrt{\pi}}\, h^+(y) \sum_{i=0}^{k} \frac{(-1)^i y^{k-i} (2\sqrt{\varepsilon z})^i}{(k-i)!\, i!} \int_{y/2\sqrt{\varepsilon z}}^{\infty} \xi^i e^{-\xi^2}\, d\xi,$$

$$v_k^-(z,y,\varepsilon) = \frac{\psi^{(k)}(-0)}{\sqrt{\pi}}\, h^-(y) \times$$

$$\times \left\{ \sum_{i=1}^{k} \frac{(-1)^i (y)^{k-i} (2\sqrt{\varepsilon z})^i}{i!\,(k-i)!} \int_{y/2\sqrt{\varepsilon z}}^{\infty} \xi^i e^{-\xi^2}\, d\xi - y^k \int_{-\infty}^{y/2\sqrt{\varepsilon z}} e^{-\xi^2}\, d\xi \right\}$$

have order $\sqrt{\varepsilon}^{\,k}$ uniformly with respect to y for $(y,z) \in D'_R$, while $v_{0\varepsilon}(z,y)$ has order $\sqrt{\varepsilon}$ for every $y \neq 0$ (here $a = 1,\ c' = 0$ has been put).

Let the coefficients b and c of equation (1) be functions of x and t. Construct a function $\tilde{a}(z,y)$ of the same smoothness as $a(z,y)$, equal to $a(z,y)$ for $|y| \geq 2\varepsilon_0$ and to $a(0,z)$ for $|y| \leq \varepsilon_0$, $\varepsilon_0 > 0$. Analogously construct $\tilde{c}'(z,y)$ from the function $c'(z,y)$. Then, by virtue of condition c), $a(z,y) = \tilde{a}(z,y) + \alpha(z,y,\varepsilon_0)$, $c'(z,y) = \tilde{c}'(z,y) + \beta(z,y,\varepsilon_0)$, where $\alpha = \beta \equiv 0$ for $|y| \geq 2\varepsilon_0$, $\alpha = O(\varepsilon_0)$, $\beta = O(\varepsilon_0)$ for $|y| < 2\varepsilon_0$. Denoting

$$\widetilde{L}_\varepsilon u^\varepsilon(z,y) = \varepsilon \tilde{a}(z,y) u^\varepsilon_{yy} - u^\varepsilon_z + \tilde{c}'(z,y) u^\varepsilon, \qquad h(\varepsilon,\alpha,\beta,u^\varepsilon) = \varepsilon \alpha u^\varepsilon_{yy} + \beta u^\varepsilon,$$

we write equation $(1')$ in the form

$$L_\varepsilon u^\varepsilon(z,y) = \widetilde{L}_\varepsilon u^\varepsilon(z,y) + h(\varepsilon,\alpha,\beta,u^\varepsilon).$$

It can be shown that for the solution $u^\varepsilon(z,y)$ of problem $(1')$, $(2')$ the representation

$$u^\varepsilon(z,y) = u_0^\varepsilon(z,y) + \sum_{k=1}^{n} u_k^\varepsilon(z,y) + O\Big(\sqrt{\varepsilon_0^{n+1}}\Big)\,, \tag{4}$$

holds.

where $u_0^\varepsilon(z,y)$ is the solution of the equation $\widetilde{L}_\varepsilon u_0^\varepsilon(z,y) = 0$ with $u_0^\varepsilon(z,y)\big|_{z=0} = \psi(y)$, and $u_k^\varepsilon(z,y)$ are determined recursively as solutions of the initial-value problems

$$\widetilde{L}_\varepsilon u_k^\varepsilon(z,y) = -h(\varepsilon,\alpha,\beta,u_{k-1}^\varepsilon), \qquad u_k^\varepsilon\big|_{z=0} = 0, \qquad k = 1,2,\dots,n,$$

and in this case $u_k^\varepsilon(z,y) = O(\varepsilon_0^k)$.

Theorem 1. If conditions a), b), c) are satisfied, then:

1) the solution $u^\varepsilon(x,t)$ of problem (1), (2) converges as $\varepsilon \to 0$ everywhere in D_∞, except for the points of $l(0,0)$, and converges uniformly in D_R, for any R, outside any neighborhood of $l(0,0)$, to the solution $v(x,t)$ of problem $(1^0), (2^0)$;
2) near the characteristic $l(0,0)$, for the solution $u^\varepsilon(x,t)$ in the variables z, y, representation (4) is valid, where $u_0^\varepsilon(z,y)$ has the form (3), and $u_k^\varepsilon(z,y)$ can be obtained by the recurrent method described above; moreover,

$$\varepsilon_0 = \delta(\varepsilon) = \sqrt{\varepsilon}\,\varkappa(\varepsilon); \qquad \lim_{\varepsilon\to 0}\varkappa(\varepsilon) = \infty; \qquad \lim_{\varepsilon\to 0}\delta(\varepsilon) = 0; \qquad u_k^\varepsilon(z,y) = O(\delta^k(\varepsilon)).$$

The proof of this theorem is based on the localization principle (see below). Let (x_1,t_1) be an arbitrary point of D_∞. Introduce the function $\psi_\delta(x)$, equal to $\psi(x)$ for $x_0-\delta \le x \le x_0+\delta$ and equal to zero for $x > x_0+\delta$, $x < x_0-\delta$, where x_0 is the point of intersection of the characteristic $l(x_1,t_1)$ with $t=0$.

Theorem 2 (localization principle). If $u_\delta^\varepsilon(x,t)$ is a solution of equation (1) satisfying the condition $u_\delta^\varepsilon(x,t)\big|_{t=0} = \psi_\delta(x)$, then

$$u^\varepsilon(x_1,t_1) - u_\delta^\varepsilon(x_1,t_1) = O\left(\frac{1}{\varkappa(\varepsilon)}e^{-\varkappa^2(\varepsilon)/2}\right), \tag{5}$$

where $\delta(\varepsilon) = B\sqrt{\varepsilon}\,\varkappa(\varepsilon) \to 0$ as $\varepsilon \to 0$; $\lim_{\varepsilon\to 0}\varkappa(\varepsilon) = \infty$ (for example, $\varkappa(\varepsilon) = \ln\frac{1}{\varepsilon}$); B is a constant.

Denote

$$\varphi(z,y) = \int_0^y \frac{d\eta}{\sqrt{a(z,\eta)}} \qquad (\varphi(z,y) \to \pm\infty \text{ as } y \to \pm\infty).$$

Jevre (^{2}) and Feller (^{3}) constructed the fundamental solution $U^\varepsilon(z,y;\tau,\xi)$ of equation $(1')$

$$U^\varepsilon(z,y;\tau,\xi) = U_0^\varepsilon(z,y;\tau,\xi) + \sum_{n=1}^{\infty} U_n^\varepsilon(z,y;\tau,\xi),$$

where

$$U_0^\varepsilon(z,y;\tau,\xi) = \exp\left\{-\frac{(\varphi(z,y)-\varphi(\tau,\xi))^2}{4\varepsilon(z-\tau)}\right\} / \, 2\sqrt{\varepsilon(z-\tau)a(\tau,\xi)},$$

$$U_{n+1}^\varepsilon(z,y;\tau,\xi) = \int_0^z dp \int_{-\infty}^{\infty} \left\{\sqrt{\varepsilon}\,\lambda_\varepsilon(p,q)\, U_{nq}^\varepsilon(p,q;\tau,\xi) - c'(p,q)\, U_n^\varepsilon(p,q;\tau,\xi)\right\} U_0^\varepsilon(z,y;p,q)\, dq,$$

$$\lambda_\varepsilon(p,q) = \sqrt{\varepsilon}\, a(p,q)/2 + \sqrt{a(p,q)}\,\varphi_z.$$

In this case the solution $u^\varepsilon(z,y)$ is written in the form

$$u^\varepsilon(z,y) = \int_{-\infty}^{\infty} \psi(\xi)\, U^\varepsilon(z,y;0,\xi)\, d\xi. \tag{6}$$

One can prove the inequality

$$\left|\sum_{n=1}^{\infty} U_n^\varepsilon(z,y;\tau,\xi)\right| < \sum_{n=1}^{\infty} d_n \left(\varepsilon a(\tau,\xi)(z-\tau)\right)^{-1/2} \exp\left\{-\frac{|\varphi(z,y)-\varphi(\tau,\xi)|^2}{8\varepsilon(z-\tau)}\right\}, \tag{7}$$

where

$$d_n = 2^{n-1}K^{2n+1}\sqrt{(z-\tau)^{n-1}}/\Gamma(n/2); \qquad 2\sqrt{z-\tau} < K; \qquad [a(\tau,\xi)]^{-1/2} < K;$$

$$|\lambda_\varepsilon(z,y)| + |c'(z,y)| < K;$$

K is a constant.

Let now the point $p(x_1, t_1)$, when x, t are replaced by y, z, go into $P(z_1, y_1)$; then $l(x_1, t_1)$ goes into the straight line $y = y_1$, and the point $p_0(x_0, 0)$ into $P_0(0, y_1)$. And let, when z, y are replaced by $z, \varphi(z, y)$, respectively, $P(z_1, y_1)$ go into $Q(z_1, \varphi(z_1, y_1))$, and $P_0(0, y_1)$ into $Q_0(0, \varphi(0, y_1))$. Then the δ_1-neighborhood of $Q_0(0, \varphi(0, y_1))$ in the $z = 0$ plane of (z, φ) will correspond to the δ_2-neighborhood of the point $P_0(0, y_1)$ in the $z = 0$ plane of (z, y), to which, in turn, there will correspond the δ_3-neighborhood of $p_0(x_0, 0)$ on the straight line $t = 0$. It can be shown that $\delta_2 \leqslant \delta_1$.

Construct the function $\psi_{\delta_1}(y)$, $\delta_1 = 2\sqrt{\varepsilon}\,\chi(\varepsilon)$, and find the solution $u^{\varepsilon}_{\delta_1}(z, y)$ of equation $(1')$ under the condition

$$u^{\varepsilon}_{\delta_1}(z, y)\Big|_{z=0} = \psi_{\delta_1}(y).$$

Then from (6) and (7), making the substitution $\varphi(z_1, y_1) - \varphi(0, \xi) = 2\sqrt{\varepsilon z_1}\,\eta$, we obtain

$$\left|u^{\varepsilon}(z_1, y_1) - u^{\varepsilon}_{\delta_1}(z_1, y_1)\right| < 2M_1(1+S)\int_{\chi(\varepsilon)}^{\infty} e^{-\xi^2/2}\,d\xi = O\left(\frac{1}{\chi(\varepsilon)} e^{-\chi^2(\varepsilon)/2}\right), \quad (8)$$

where $S = (d_1 + d_2 + d_3 + \cdots)/2$; $|\psi| < M_1$; M_1 is a constant. From (8) follows the validity of (5), since one can always ensure that $\delta \geqslant \delta_3$.

It is not difficult to see that the result obtained also holds in the case when in (1), before u^{ε}_{xx}, there stands a coefficient $\tilde{a}(x, t) \geqslant \rho > 0$, satisfying condition b).

It can be proved that, as $\varepsilon \to 0$, the behavior of the solution of the boundary-value problem for equation (1), both for a finite interval and for a half-line inside the domain of definition of the solution, is analogous to the behavior of the solution of the Cauchy problem, and the corresponding asymptotics can be obtained.

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