AN ESTIMATE OF THE SUM OF DEFECTS OF A MEROMORPHIC FUNCTION OF ORDER LESS THAN ONE

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Fig. 1

Figure 1: Fig. 1

Abstract

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MATHEMATICS

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AN ESTIMATE OF THE SUM OF DEFECTS OF A MEROMORPHIC FUNCTION OF ORDER LESS THAN ONE

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Let $\delta(a)$ denote the Nevanlinna defect at the point a of a function $f(z)$, meromorphic in the finite z-plane $(^1)$. R. Nevanlinna showed that for meromorphic functions of nonintegral order ρ, for any $a \neq b$, $\delta(a)+\delta(b)$ is less than a certain constant, which is less than 2 and depends only on ρ, and gave estimates $(^2)$. For the case $0 \leq \rho < 1$, Shah $(^3)$, refining Nevanlinna' s estimates, showed that $\delta(a)+\delta(b) \leq 1+\rho$.

The purpose of the present note is to prove the inequality

$$\delta(a)+\delta(b) \leq \begin{cases} 1, & 0 \leq \rho \leq 1/3, \\ 2-2^{\rho}\sqrt{\pi}\,\Gamma\left(1-\dfrac{\rho}{2}\right)\Gamma^{-1}\left(\dfrac{1-\rho}{2}\right), & 1/3 < \rho < 1, \end{cases} \tag{1}$$

which, as is not difficult to verify, is a sharper estimate than Shah' s estimate. The estimate (1) is probably not the best possible (for $1/3 < \rho < 1$), but, in any case, it does not deviate very greatly from it, since for every entire function of order $0 \leq \rho < 1/2$, $\delta(\infty)+\delta(0) = 1$, and for the entire function

$$\prod_{n=1}^{\infty}(1 - zn^{-1/\rho})$$

of order $1/2 \leq \rho < 1$, $\delta(\infty)+\delta(0) = 2-\sin\pi\rho$. The graphs of the corresponding functions are shown in Fig. 1.

Fig. 1

Let us have a meromorphic function $w = f(z)$ of order ρ, $0 \leq \rho < 1$. Without loss of generality, we may restrict ourselves to estimating $\delta(0)+\delta(\infty)$ and assume that $f(0) = 1$. Then

$$f(z) = \prod_{\nu} \left(1 - z a_{\nu}^{-1}\right) \cdot \prod_{\mu} \left(1 - z b_{\mu}^{-1}\right)^{-1},$$

where the series $\sum_{\nu} |a_{\nu}|^{-\lambda}$, $\sum_{\mu} |b_{\mu}|^{-\lambda}$ and the integrals

$$\int_0^{\infty} N(r, 0) r^{-\lambda-1}\, dr,$$

$$\int_0^{\infty} N(r, \infty) r^{-\lambda-1}\, dr$$

converge for all λ, $\rho < \lambda \leq 1$; from this will follow the convergence of all the series and integrals that we shall encounter below.

$$m(r, 0) = \frac{1}{2\pi} \int_0^{2\pi} \ln^{+} \left| \frac{1}{f(re^{i\varphi})} \right| d\varphi \leq \sum_{\nu} \frac{1}{2\pi} \int_0^{2\pi} \ln^{+} \left| \frac{a_{\nu}}{re^{i\varphi} - a_{\nu}} \right| d\varphi + \\ + \sum_{\mu} \frac{1}{2\pi} \int_0^{2\pi} \ln^{+} \left| \frac{re^{i\varphi} - b_{\mu}}{b_{\mu}} \right| d\varphi = \sum_{\nu} u_{\nu}(r) + \sum_{\mu} U_{\mu}(r). \tag{2}$$

Denote by c^{ν} (C^{μ}) the disk $|z - a_{\nu}| < |a_{\nu}|$ ($|z - b_{\mu}| > |b_{\mu}|$); by γ_r^{ν} (Γ_r^{μ}) the arcs of the circle $|z| = r$ lying in c^{ν} (C^{μ}); and by $\omega_{\nu}(r)$ ($\Omega_{\mu}(r)$) the radian measure of the arc of the circle $|z - a_{\nu}| = |a_{\nu}|$ ($|z - b_{\mu}| = |b_{\mu}|$) lying inside the disk $|z| < r$.

A simple calculation gives that $\omega_{\nu}(r) = 4 \arcsin(2^{-1} |a_{\nu}|^{-1} r)$ for $0 \leq r \leq 2|a_{\nu}|$; $\Omega_{\mu}(r) = 4 \arcsin(2^{-1} |b_{\mu}|^{-1} r)$ for $0 \leq r \leq 2|b_{\mu}|$; $\Omega_{\mu}(r) = 2\pi$ for $2|b_{\mu}| \leq r < \infty$.

Further,

$$\frac{du_{\nu}(r)}{d \ln r} = \frac{1}{2\pi} \int_{\gamma_r^{\nu}} \frac{\partial}{\partial r} \ln \left| \frac{a_{\nu}}{re^{i\varphi} - a_{\nu}} \right| r\, d\varphi = -\frac{1}{2\pi} \int_{\gamma_r^{\nu}} d \arg(z - a_{\nu}).$$

Consequently,

$$\frac{du_{\nu}}{d \ln r} = \frac{\omega_{\nu}(r)}{2\pi} \quad \text{for } 0 \leq r < |a_{\nu}|;$$

$$\frac{du_{\nu}}{d \ln r} = \frac{\omega_{\nu}(r) - 2\pi}{2\pi} = 0 \quad \text{for } |a_{\nu}| < r \leq 2|a_{\nu}|;$$

$$\frac{du_{\nu}}{d \ln r} \quad \text{for } 2|a_{\nu}| < r < \infty.$$

$$u_{\nu}(r) = \begin{cases} \dfrac{2}{\pi} \displaystyle\int_0^{r} \arcsin \frac{r}{2|a_{\nu}|} \frac{dr}{r} - \ln^{+} \frac{r}{|a_{\nu}|}, & 0 \leq r \leq 2|a_{\nu}|, \\ 0, & 2|a_{\nu}| \leq r < \infty. \end{cases} \tag{3}$$

$$u_\nu(r) + \ln^+ \frac{r}{|a_\nu|} = \begin{cases} \dfrac{2}{\pi} \displaystyle\int_0^r \arcsin \frac{r}{2|a_\nu|} \frac{dr}{r}, & 0 \le r \le 2|a_\nu|, \\ \ln \dfrac{r}{|a_\nu|}, & 2|a_\nu| \le r < \infty. \end{cases} \tag{4}$$

Analogously we obtain

$$\frac{dU_\mu(r)}{d\ln r} = \frac{1}{2\pi} \int_{\Gamma_r^\mu} \frac{\partial}{\partial r} \ln \left| \frac{re^{i\varphi} - b_\mu}{b_\mu} \right| r\, d\varphi = \frac{1}{2\pi} \int_{\Gamma_r^\mu} d \arg(z - b_\mu) = \frac{\Omega_\mu(r)}{2\pi},$$

$$U_\mu(r) = \begin{cases} \dfrac{2}{\pi} \displaystyle\int_0^r \arcsin \frac{r}{2|b_\mu|} \frac{dr}{r}, & 0 \le r \le 2|b_\mu|, \\ \ln \dfrac{r}{|b_\mu|}, & 2|b_\mu| \le r < \infty. \end{cases} \tag{5}$$

Introduce the function $\chi(x) = \frac{2}{\pi} \int_0^x \arcsin \frac{x}{2} \frac{dx}{x}$ for $0 \le x \le 2$; $\chi(x) = \ln x$ for $2 < x < \infty$. It is not difficult to establish the continuity of $\chi(x)$ for $0 \le x < \infty$. We also note that, since $u_\nu(r) > 0$ for $0 < r < 2|a_\nu|$, it follows from (3) that

$$\frac{2}{\pi} \int_0^x \arcsin \frac{x}{2} \frac{dx}{x} > \ln^+ x, \qquad 0 < x < 2. \tag{6}$$

From (2), (4), (5) we obtain

$$T(r, f) = m(r, 0) + N(r, 0) = m(r, 0) + \sum_\nu \ln^+ \frac{r}{|a_\nu|} \le$$

$$\le \sum_\nu \left(u_\nu(r) + \ln^+ \frac{r}{|a_\nu|} \right) + \sum_\mu U_\mu(r) = \sum_\nu \chi\left(\frac{r}{|a_\nu|}\right) + \sum_\mu \chi\left(\frac{r}{|b_\mu|}\right). \tag{7}$$

We shall now prove the following auxiliary inequality: for $0 < a < \infty$, $0 < \lambda < 1$,

$$\varkappa(\lambda) = \Gamma\left(\frac{1-\lambda}{2}\right) 2^{-\lambda} \pi^{-1/2} \Gamma^{-1} \left(1 - \frac{\lambda}{2}\right)$$

$$\int_a^\infty \chi(x) x^{-\lambda-1}\, dx < \varkappa(\lambda) \int_a^\infty (\ln^+ x) x^{-\lambda-1}\, dx. \tag{8}$$

Similar auxiliary inequalities for obtaining various estimates have been used by many authors (see, for example, ($^{4-6}$)). By a simple, though somewhat cumbersome, calculation one verifies that

$$\int_0^\infty \chi(x)x^{-\lambda-1}\,dx = \varkappa(\lambda)\int_0^\infty (\ln^+ x)x^{-\lambda-1}\,dx. \tag{9}$$

From (6), (9), and the definition of $\chi(x)$ it follows that $\varkappa(\lambda) > 1$. Let

$$\Phi(a) = \int_a^\infty [\varkappa(\lambda)\ln^+ x - \chi(x)]x^{-\lambda-1}\,dx = \int_a^\infty [-\psi(x)]x^{-\lambda-1}\,dx.$$

$$\Phi'(a) = \psi(a)a^{-\lambda-1}.$$

Obviously, $\psi(a) > 0$ for $0 < a \le 1$. Further,

$$\psi'(a) = 2(\pi a)^{-1}\arcsin(a/2) - a^{-1}\varkappa(\lambda) < 0$$

for $1 < a \le 2$, and

$$\psi'(a) = [1 - \varkappa(\lambda)]a^{-1} < 0$$

for $2 \le a < \infty$. Consequently, for $1 < a < \infty$, ψ is strictly monotonically decreasing, and since $\psi(1) > 0$ and $\psi(2) = [1 - \varkappa(\lambda)]\ln 2 < 0$, there exists a unique point a_0, $\psi(a_0) = 0$, $1 < a_0 < 2$, such that $\psi(a) > 0$ for $0 < a < a_0$ and $\psi(a) < 0$ for $a_0 < a < \infty$. Hence $\Phi(a)$ is strictly monotonically increasing for $0 < a < a_0$ and strictly monotonically decreasing for $a_0 < a < \infty$. But $\Phi(a) \to 0$ as $a \to \infty$, and $\Phi(0) = 0$ by virtue of (9); therefore $\Phi(a) > 0$ for all $a > 0$, and this is equivalent to (8).

From (8), for $\rho < \lambda < 1$, it follows that

$$\int_0^\infty \left[\sum_\nu \chi\left(\frac{r}{|a_\nu|}\right) + \sum_\mu \chi\left(\frac{r}{|b_\mu|}\right)\right] r^{-\lambda-1}\,dr <$$

$$< \varkappa(\lambda)\int_a^\infty \left(\sum_\nu \ln^+\frac{r}{|a_\nu|} + \sum_\mu \ln^+\frac{r}{|b_\mu|}\right) r^{-\lambda-1}\,dr =$$

$$= \varkappa(\lambda)\int_a^\infty [N(r,0) + N(r,\infty)]r^{-\lambda-1}\,dr.$$

Since in (10) a can be chosen arbitrarily large, there exists a sequence $r_k \to \infty$ such that

$$\sum_\nu \chi\left(\frac{r_k}{|a_\nu|}\right) + \sum_\mu \chi\left(\frac{r_k}{|b_\mu|}\right) < \varkappa(\lambda)\,[N(r_k,0) + N(r_k,\infty)].$$

By (7),

$$T(r_k, f) < \varkappa(\lambda)[N(r_k, 0) + N(r_k, \infty)];$$

$$\overline{\lim}_{r\to\infty}[N(r, 0) + N(r, \infty)]/T(r, f) \geq \varkappa^{-1}(\lambda).$$

Since $\varkappa(\lambda)$ is a continuous function, letting $\lambda \to \rho$ we obtain

$$\overline{\lim}_{r\to\infty}[N(r, 0) + N(r, \infty)]/T(r, f) \geq \varkappa^{-1}(\rho).$$

Hence it follows that $\delta(0)+\delta(\infty) \leq 2-\varkappa^{-1}(\rho)$, $0 \leq \rho < 1$. For $1/3 < \rho < 1$ this is our desired estimate (1). For $0 \leq \rho \leq 1/3$, estimate (1) follows from assertion (7): if for $0 \leq \rho < 1/2$ $\delta(a) > 1 - \cos\pi\rho$, then a is the only deficient value of $f(z)$; indeed, for $0 \leq \rho \leq 1/3$, $\max[1, 2(1 - \cos\pi\rho)] = 1$.

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Note: Figure translations are in progress. See original paper for figures.