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Abstract

Full Text

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Absolute Minimum in a Two-Dimensional Parametric Problem on a Manifold

(Presented by Academician P. S. Aleksandrov, 27 XI 1956)

A.

The existence of a surface with a prescribed contour giving an absolute minimum of a positively regular integral, for a general integrand and for surfaces in Euclidean space, was first established by A. G. Sigalov¹⁻³ (and later, by another method, by Cesari⁴). For surfaces in more complicated spaces, the only general result known to us was obtained by Morrey⁵, who proved the existence of an absolute minimum of area in Riemannian manifolds of a very general type; these manifolds include, in particular, all compact manifolds and Euclidean spaces.

We prove the existence of an absolute minimum for an arbitrary positively regular problem on manifolds \mathfrak{M} of Morrey type. For compact manifolds the solution is obtained without any metric restrictions; for noncompact ones, corresponding conditions at infinity are introduced. In the case where \mathfrak{M} is Euclidean space, Sigalov's theorem follows from our results; in the case of the Plateau problem, Morrey's theorem follows. Instead of Courant's method, which underlies Morrey's work, we apply A. G. Sigalov's method. In this way the solution is obtained as the limit of a uniformly convergent minimizing sequence, something Morrey did not achieve.

In this note we restrict ourselves to the case of a single boundary contour and do not aim at minimal smoothness assumptions concerning \mathfrak{M} .

B.

Let \mathfrak{M} be a differentiable manifold whose local coordinates are related by transformations $y^i = y^i(x^1, \dots, x^n)$ ($i = 1, 2, \dots, n$), where the y^i have first derivatives satisfying a Lipschitz condition (a manifold of smoothness C'_1).

If a metric tensor g_{ij} is given on \mathfrak{M} , then, following Morrey, the metric is called *homogeneously regular* if the g_{ij} have the same smoothness as \mathfrak{M} , and for each point $p \in \mathfrak{M}$ there exists a neighborhood $U \ni p$ and in U a local coordinate system (x^1, \dots, x^n) such that: 1) $-1 < x^i < 1$ ($i = 1, 2, \dots, n$); 2) the point p has zero coordinates; 3) for all $x \in U$ the inequalities

$$m \sum_{i=1}^n \xi^{i2} \leq g_{ij}(x) \xi^i \xi^j \leq M \sum_{i=1}^n \xi^{i2}, \quad (1)$$

hold, where the ξ^i are arbitrary numbers; $M > m > 0$ are numbers depending only on \mathfrak{M} and on the metric g_{ij} . We shall call neighborhoods U and coordinates x^i of the indicated type *basic*.

Let a function $F(p, \mathfrak{B}_p)$ of a point p and of a simple bivector \mathfrak{B}_p at the point p be defined on \mathfrak{M} (⁶, p. 124). In coordinates F is written in the form

$$F(x^1, \dots, x^n; A^{12}, \dots, A^{n-1n}) = F(x, A),$$

where

$$A^{ij} = \begin{vmatrix} a^i & a^j \\ b^i & b^j \end{vmatrix} \quad (i \neq j)$$

are the components of \mathfrak{B}_p .

Assume that F satisfies the conditions:

I. **Continuity.** $F(x^1, \dots, x^n; A^{12}, \dots, A^{n-1n})$ is a continuous function of its arguments.

II. **Positive homogeneity.** For $k > 0$,

$$F(x, kA) = kF(x, A).$$

III. **Positive regularity.**

$$F(x, A_1 + A_2) \leq F(x, A_1) + F(x, A_2).$$

IV. **Uniform regularity.** On \mathfrak{M} one can introduce a uniformly regular Riemannian metric such that in every fundamental neighborhood U the inequalities

$$m\|A\| \leq F(x, A) \leq M\|A\|, \quad (2)$$

hold, where $\|A\|^2 = \sum_{i \neq j} (A^{ij})^2$, $M > m > 0$ are numbers depending only on \mathfrak{M} , g_{ij} , and F .

For compact \mathfrak{M} condition IV is always fulfilled by virtue of Whitney's results on the metrization of differentiable manifolds (⁷). For Euclidean space (2) passes into Sigalov's condition (³). In the case of Plateau's problem (2) follows from (1), so that in Morrey's case IV is fulfilled.

In the special case of a Riemannian area on \mathfrak{M} we shall denote F by F_R .

B. Denote by L_2^T the class of vector-functions $x(u, v) = (x^1(u, v), \dots, x^n(u, v))$ satisfying the following conditions: 1) $x^i(u, v)$ are continuous in the square $K[0 \leq u, v \leq 1]$; 2) $x^i(u, v)$ are absolutely continuous in Tonelli's sense⁽⁸⁾; 3) the derivatives x_u^i, x_v^i are summable in the square K .

We shall call an **admissible surface** $p = p(u, v)$, $p \in \mathfrak{M}$, if in local coordinates of \mathfrak{M} $p(u, v)$ has the representation $x = x(u, v)$, where $x(u, v) \in L_2^T$. For an admissible surface $p(u, v)$ put

$$(p, K, F) = \sum_{\mathcal{N}_k} \iint_{\mathcal{N}_k} F \left(x^i(u, v); \begin{vmatrix} x_u^i & x_v^i \\ x_u^j & x_v^j \end{vmatrix} \right) du dv,$$

where $\{\mathcal{N}_k\}$ is a partition of K into polygons \mathcal{N}_k such that each surface $p(\mathcal{N}_k)$ lies in one of the local coordinate neighborhoods of \mathfrak{M} . (p, K, F) depends neither on the partition $\{\mathcal{N}_k\}$ nor on the parametrization p (in the class L_2^T).

C. Theorem. *Let on a differentiable manifold \mathfrak{M} of smoothness C_1' there be given a function $F(p, \mathfrak{B}_p)$ satisfying conditions I–IV. Let Γ be a simple closed contour on \mathfrak{M} . If there exists at least one admissible surface with boundary Γ for which (p, K, F) is finite, then there exists an admissible surface p_0 with boundary Γ , giving the functional (p, K, F) the least value in the class of all admissible surfaces with boundary Γ .*

D. The proof of the theorem is based on the following lemmas.

Lemma 1. *There exist numbers $B > b > 0$, depending only on \mathfrak{M} , F , g_{ij} , such that in every fundamental neighborhood U*

$$bF_R(x, A) \leq F(x, A) \leq BF_R(x, A).$$

Lemma 2. *There exists a number $N > 0$, depending only on \mathfrak{M} , F , g_{ij} , such that every closed contour γ of Riemannian length $l(\gamma)$, lying in any fundamental neighborhood U , is the boundary of an admissible surface p_γ lying in U , for which*

$$(p_\gamma, K, F) \leq N[l(\gamma)]^2.$$

Lemma 3. *Let U be a fundamental neighborhood with coordinates x^1, \dots, x^n ;*

$$\varphi(x) = \left[\sum_{i=1}^n x^{i2} \right]^{1/2};$$

p_1 an admissible surface with boundary γ , lying in the sphere $\varphi(x) < \frac{1}{4}$, and representing a polyhedron in the region $\frac{1}{4} \leq \varphi(x) \leq \frac{3}{4}$. Let G_c , $\frac{1}{3} \leq c \leq \frac{2}{3}$, be the connected component of the set of points ($\varphi < c$) of the square K , containing the fixed point $(u_0, v_0) \in K$. Denote

through γ_c the outer boundary contour G_c , and by \widetilde{G}_c the part of K bounded by γ_c , and put

$$Q(c) = \frac{(p_1, \tilde{G}_c, F)}{l^1(\gamma_c)^2}.$$

Then there exists a c_0 , $1/3 \leq c_0 \leq 2/3$, such that

$$Q(c_0) > \frac{T}{(p_1, K, F)},$$

where $T > 0$ depends only on \mathfrak{M}, F, g_{ij} .

Lemma 4. Let p_1 be the part of the admissible surface p corresponding to the rectangle $K_1 \subset K$ and satisfying the conditions of Lemma 3; let γ be the boundary of p_1 . Suppose (in the notation of Lemma 3) that

$$l(\gamma) < \frac{1}{N} \sqrt{\frac{T}{2}}.$$

Then one can replace p_1 by a surface p' , lying in the ball $\mathfrak{S}(x) < 2/3$ and bounded by the same contour γ , in such a way that for the surface p' obtained from p

$$(p', K, F) < (p, K, F).$$

Lemma 5. There exists a minimizing sequence $p = p_n(u, v)$ for the problem F such that:

- 1) the Dirichlet integrals

$$\frac{1}{2} \iint (E + G) du dv$$

are uniformly bounded for all $p_n(u, v)$ (here, in local coordinates, $E = g_{ij}(x)x_u^i x_u^j$, $G = g_{ij}(x)x_v^i x_v^j$);

- 2) the boundary curves of the surfaces $p_n(u, v)$ converge uniformly to the curve $\Gamma : p_0(u, v)$.

The proof is based on Carathéodory's theorem on the conformal mapping of surfaces of class C_1' ⁽⁹⁾, pp. 117-121, and on the generalized theorem of McShane ⁽¹⁰⁾.

Lemma 6. For every $\varepsilon > 0$ and every n , beginning with some N , one can construct Young lattices ⁽¹¹⁾ $\mathfrak{A}_n, \mathfrak{B}_n, \mathfrak{C}_n$, with a number of straight lines independent of n , such that:

- 1) the open rectangles q of the lattices $\mathfrak{A}_n, \mathfrak{B}_n, \mathfrak{C}_n$, for each n , form a covering of K , moreover such that every set of diameter less than some $\delta > 0$ lies in one of the elements of the covering;
- 2) the curves on $p_n(u, v)$ ($n = N, N + 1, \dots$), corresponding to the boundaries of analogously situated rectangles q , have lengths less than ε , and converge in the sense of Fréchet to a certain closed curve.

The proof is based on Hilbert's compactness theorem.

Lemma 7. The minimizing sequence of Lemma 5 can be replaced by another, denoted likewise by $p = p_n(u, v)$, for which the following condition is fulfilled: whatever rectangle $q' \subset K$, lying in one of the rectangles q of Lemma 6 at a distance greater than $\delta/2$ from the boundary of q , the Riemannian distance from the points of the surface $p_n(q')$ to the boundary of this same surface is less than $\lambda(\sigma)$, where σ is the area of q' , and

$$\lim_{\sigma \rightarrow 0} \lambda(\sigma) = 0.$$

The proof is obtained by applying Lemma 4 and then by repeated application of the redefinitions of A. G. Sigalov ⁽³⁾ in all the principal neighborhoods U containing the surfaces $p_n(q)$.

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