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Abstract

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ON A CLASS OF EQUATIONS WITH POSITIVE OPERATORS

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In the present article a new definition of concave operators is proposed. It turns out that for equations with such operators many theorems are valid which were previously established for more special classes of equations ([1,3–5,7]). We use the terminology of the theory of cones of M. G. Krein ([4,6]). We note that the main result of the article (Theorem 1) is obtained by topological methods.

1. Below, K and K_1 denote two cones in a real Banach space E, and it is assumed that $K \subset K_1$. The sign $\ll$ denotes the semi-ordering generated by the "larger" cone K_1: one writes $X \ll Y$ if $y - x \in K_1$. By K_r is denoted the intersection of the "smaller" cone K with the ball $\|\varphi\| \leq r$. It is assumed that $\|x\| \leq \|y\|$ if $\theta \ll x \ll y$. By A is denoted an operator (in general, nonlinear) defined on K_r. We shall say that the operator A is **positive** if $AK_r \subset K$, and that the operator A is **monotone** if from $\varphi \ll \psi$ $(\varphi, \psi \in K_r)$ it follows that $A\varphi \ll A\psi$. We emphasize that the concepts of positivity and monotonicity have been introduced with the aid of different cones. By u_0 we denote some fixed nonzero element of K.

Definition. A positive and monotone operator A will be called $\{K_1, u_0\}$-concave if:

1) to each $\varphi \in K_r$ $(\varphi \neq 0)$ there correspond numbers $\alpha, \beta > 0$ such that

$$\alpha u_0 \ll A\varphi \ll \beta u_0;$$

2) if $\varphi \in K_r$ and $\varphi \gg \gamma u_0$ $(\gamma > 0)$, then

$$At\varphi \gg tA\varphi, \qquad At\varphi \neq tA\varphi \quad (0 < t < 1); \tag{1}$$

3) for each pair of elements $\varphi_1, \varphi_2 \in K_r$ $(\varphi_1, \varphi_2 \gg \varkappa u_0,\ \varkappa > 0,\ \varphi_1 - \varphi_2 \in K_1)$, from $\varphi_1 \gg t\varphi_2$ $(t > 0,\ \varphi_1 \neq t\varphi_2)$ it follows that

$$A\varphi_1 - tA\varphi_2 \gg \delta u_0 \qquad (\delta > 0). \tag{2}$$

2. The equation

$$A\varphi = \lambda\varphi \tag{3}$$

with $\{K_1, u_0\}$-concave operators has important properties, analogous to those established for other equations in ([1,3–5,7]). We formulate the most important properties in the form of a theorem.

Nonzero solutions of equation (3) are called **eigenvectors**, and the corresponding values λ are called **eigenvalues**.

Theorem 1. *Let the operator A ($A\theta = \theta$) be completely continuous and $\{K_1, u_0\}$-concave.*

Then the following assertions hold:

1) The eigenvectors of the operator A form in K_r a continuous branch* of length r.
2) The corresponding eigenvalues completely fill a certain interval.
3) To each eigenvalue λ there corresponds a unique nonzero solution $\varphi(\lambda)$ of equation (3) in K_r.
4) From $\lambda_1 < \lambda_2$ it follows that $\varphi(\lambda_2) \ll \varphi(\lambda_1)$.
5) The function $\varphi(\lambda)$ is strongly continuous in λ.

The endpoints of the interval of eigenvalues can be found by the same method as in (4).

3. Assertions analogous to Theorem 1 were first established by P. S. Uryson for an integral equation of the form

$$\varphi(x) = \lambda \int_0^1 \mathcal{K}(x, s, \varphi(s))\, ds.$$

The investigations of P. S. Uryson served as the starting point for the creation of the theory of u_0-concave operators ([4,5]). Subsequently, assertions analogous to Theorem 1 were established ([1]) for the nonlinear integral equation

$$\lambda\varphi(x) = \rho(x) \int_0^1 G(x, s)\varphi(s)\, ds \times \left[1 - \left(\int_0^1 \frac{\partial G(x, s)}{\partial x}\varphi(s)\, ds\right)^2\right]^{1/2},$$

arising in the study of the longitudinal bending of rods of variable stiffness. This equation gave occasion for singling out the class of u_0-monotone operators ([4]), for which assertions of the type of Theorem 1 are also valid. The classes of u_0-concave and u_0-monotone operators (for definitions see ([4])) turned out not to contain one another. In connection with this there arose the problem of finding broader classes of equations for which Theorem 1 is valid. In a certain sense this problem is solved, since the following theorem holds.

Theorem 2. *Every u_0-concave and every u_0-monotone operator is a $\{K_1, u_0\}$-concave operator.*

4. The application of Theorem 1 to the study of concrete equations requires verification of the conditions of $\{K_1, u_0\}$-concavity. By Theorem 2, examples of $\{K_1, u_0\}$-concave operators will be the operators studied in ([1,3–5,7]).

As a new example, consider the operator A, defined in the space C_n of vector-functions continuous on the interval $[0,1]$, $\vec{\varphi}(x) = \{\varphi_1(x), \varphi_2(x), \dots, \varphi_n(x)\}$, by the formula

$$A\vec{\varphi}(x) = \{A_1\vec{\varphi}(x), A_2\vec{\varphi}(x), \dots, A_n\vec{\varphi}(x)\}, \tag{4}$$

where

$$A_i\vec{\varphi}(x) = \int_0^1 \mathcal{K}_i(x, s, \varphi_1(s), \varphi_2(s), \dots, \varphi_n(s))\, ds. \tag{5}$$

To transfer the theory of P. S. Uryson to systems

$$A\vec{\varphi}(x) = \lambda\vec{\varphi}(x) \tag{6}$$

one may use Theorem 1. For this–

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* By a continuous branch, following M. A. Krasnosel' skii ([4]), we mean such a set $\mathfrak{A}$ of eigenvectors that $\mathfrak{A} \cap \Gamma \neq 0$ for the boundary Γ of any domain of the space E containing zero θ and contained in the ball $\|x\| \leq r$.

one must find conditions for $\{K_1, u_0\}$-concavity of the operator $A\vec{\varphi}(x)$ with respect to certain cones K and K_1 and to a certain element u_0.

As K_1 we shall consider the set of nonnegative vector-functions of the space C_n. As the cone K we consider the set $K = K(u_0, c)$ of vector-functions $\vec{\psi}(x) = \{\psi_1(x), \psi_2(x), \dots, \psi_n(x)\}$, whose components satisfy the conditions

$$\alpha u_i^{(0)}(x) \leqslant \psi_i(x) \leqslant \beta u_i^{(0)}(x) \quad (i = 1, 2, \dots, n), \tag{7}$$

where $\alpha, \beta > 0$; $\beta \leqslant c_i\alpha$; $u_0(x) = \{u_1^{(0)}(x), u_2^{(0)}(x), \dots, u_n^{(0)}(x)\}$ is a fixed element of K_1 with components not identically equal to zero; $c = \{c_1, c_2, \dots, c_n\}$ $(c_i \geqslant 1)$ is a fixed vector, and the element $\vec{\theta}(x) = \{0, 0, \dots, 0\}$.

Assume that the nonnegative functions $\mathcal{K}_i(x, s, u_1, u_2, \dots, u_n)$ $(0 \leqslant x,\ s \leqslant 1,\ 0 \leqslant u_1,\ u_2, \dots, u_n < \infty)$ $(i = 1, 2, \dots, n)$ are continuous and satisfy the conditions:

1) $\mathcal{K}_i(x, s, u_1, u_2, \dots, u_n)$ are nondecreasing, and

$$\frac{\mathcal{K}_i(x, s, u_1, u_2, \dots, u_n)}{\sqrt{u_1^2 + u_2^2 + \dots + u_n^2}}$$

are nonincreasing functions of the variables $u_1, u_2, \dots, u_n$.

2) To every set of numbers $u_1, u_2, \dots, u_n \geqslant 0$, for which the functions $\mathcal{K}_i(x, s, u_1, u_2, \dots, u_n) \not\equiv 0$, and to every $t \in (0, 1)$, there correspond such continuous nonnegative functions $\gamma_i(x)$ that

$$\frac{\mathcal{K}_i(x, s, tu_1, tu_2, \dots, tu_n)}{t\sqrt{u_1^2 + u_2^2 + \dots + u_n^2}} - \frac{\mathcal{K}_i(x, s, u_1, u_2, \dots, u_n)}{\sqrt{u_1^2 + u_2^2 + \dots + u_n^2}} \geqslant \gamma_i(s)\, u_i^{(0)}(x),$$

where $\gamma_i(x)$ is positive almost at all those points where $u_i^{(0)}(x) > 0$.

3) There exist such nonnegative continuous functions $\alpha(s)$ and $\beta(s)$ that for every set $u_1, u_2, \dots, u_n \geqslant 0$, for which $\mathcal{K}_i(x, s, u_1, u_2, \dots, u_n) \not\equiv 0$, the inequalities

$$\alpha(s) u_i^{(0)}(x) \leqslant \frac{\mathcal{K}_i(x, s, u_1, u_2, \dots, u_n)}{\sqrt{u_1^2 + u_2^2 + \dots + u_n^2}} \leqslant \beta(s) u_i^{(0)}(x) \quad (i = 1, 2, \dots, n),$$

hold, where $\alpha(x) u_i^{(0)}(x) \not\equiv 0$ almost at all points where $u_i^{(0)}(x) \not\equiv 0$, and $\beta(s) \leqslant c_i \alpha(s)$ $(i = 1, 2, \dots, n)$.

4) For each i $(i = 1, 2, \dots, n)$ there exists its own point $s_0 \in [0, 1]$, at which

$$\mathcal{K}_i(x, s_0, u_1^{(0)}(s_0), u_2^{(0)}(s_0), \dots, u_n^{(0)}(s_0)) \not\equiv 0, \quad u_i^{(0)}(s_0) \not\equiv 0.$$

Under these assumptions the operator (6) is $\{K_1, u_0\}$-concave.

The theory of u_0-concave operators makes use, in particular, of many properties of special classes of linear positive operators. These properties for linear integral operators were already found by Jentzsch ([2]). The corresponding theorems were extended to the case of abstract positive linear operators by M. G. Krein and M. A. Rutman ([6]), and to the special case by M. A. Krasnosel' skii and L. A. Ladyzhenskii ([5]).

Our constructions required the study of a broader class of linear positive operators.

We shall say that a linear operator A, leaving invariant the "smaller" cone K, is u_0-strongly positive with respect to the "larger" cone K_1 $(K_1 \supset K)$, if there is an element $u_0 \in K$ $(u_0 \neq \theta)$ such that for every $\varphi \in K_1$ $(\varphi \neq \theta)$ there exist a natural number n and numbers $\alpha, \beta > 0$ such that

$$\alpha u_0 \ll A^n \varphi \ll \beta u_0.$$

Theorem 3. *Let A be a monotone linear completely continuous operator that is u_0-strongly positive with respect to K_1. Suppose that for every $\varphi \in E$ one can indicate a natural number $p > 0$ and a number $l > 0$ such that $lA^p\varphi \ll u_0$. Then the operator A has a simple positive eigenvalue, which is greater than the absolute value of all other eigenvalues and to which there corresponds an eigenvector $\varphi \in K$.*

This theorem is a direct generalization of known propositions on linear integral operators with positive kernels. It contains, as various special cases, Jentzsch' s theorem and some theorems from $(5, 6)$.

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Note: Figure translations are in progress. See original paper for figures.