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Abstract

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ON A UNIQUENESS THEOREM FOR THE SOLUTION OF A MIXED PROBLEM FOR SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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In the note of I. M. Gel'fand and G. E. Shilov (^{1}), a new approach is indicated to the problem of uniqueness of the Cauchy solution for systems of linear equations of the form

$$\frac{\partial u}{\partial t} = P\left(t, \frac{\partial}{\partial x}\right) u, \qquad u(x,0) = u_0(x).$$

In the present note this method is applied to establishing uniqueness of the solution of the mixed problem for certain types of systems of linear equations.

Let there be given a system of linear partial differential equations of the form

$$\frac{\partial u}{\partial t} = P\left(t, \frac{\partial^n}{\partial x^n}\right) u, \qquad n \geqslant 2 \text{ integer}, \qquad x \geqslant 0, \qquad t \geqslant 0, \tag{1}$$

where $u(x,t) = \{u_1(x,t), \dots, u_N(x,t)\}$ is an unknown vector-function; $P\left(t, \frac{\partial^n}{\partial x^n}\right)$ is a matrix of N rows and N columns whose elements are polynomials in the operator $D \equiv \partial^n/\partial x^n$ with coefficients continuously depending on t. Let nq_j be the highest order of the derivatives of $u_j(x,t)$ with respect to x in system (1). To system (1) are adjoined the initial condition

$$u(x,0) = u_0(x) \tag{2}$$

and boundary conditions of one of the following types:

$$\left.\frac{\partial^{nl_j+m} u_j}{\partial x^{nl_j+m}}\right|_{x=0} = 0, \qquad l_j = 0, 1, \dots, q_j - 1, \tag{3}$$

where m takes one (fixed) of the values $0, 1, \dots, n-1$.

Theorem. Suppose that system (1) has reduced order (^{1,2})

$$1 \leqslant p_0 < \frac{n}{n-2}. \tag{4}$$

If there exists a vector-function $u(x,t)$ satisfying system (1), conditions (2), (3), and, for every $t \geqslant 0$, the inequality

$$|u(x,t)| \leqslant A \exp\left[B|x|^{p_0'-\varepsilon}\right], \qquad p_0' = \frac{p_0}{p_0-1}, \qquad \varepsilon > 0, \tag{5}$$

then $u(x,t)$ is the unique solution of the mixed problem (1)–(3).

Proof. First we shall solve the adjoint problem in the basic space (2)Φ of entire vector-functions $\varphi(x)$ with the following properties:

$$|x^k \varphi^{(q)}(x)| \leq C_\varphi A^k B^q k^{\alpha k} q^{\beta q}, \qquad x \geq 0, \qquad k, q = 0, 1, \dots, \qquad \beta < 1; \tag{6}$$

$$\varphi^{(nj)}(0) = \varphi^{(nj+1)}(0) = \dots = \varphi^{(nj+n-m-2)}(0) = \varphi^{(nj+n-m)} = \dots$$

$$\dots = \varphi^{(nj+n-1)}(0) = 0, \qquad j = 0, 1, 2, \dots \tag{7}$$

As G. E. Shilov showed in (3), the space S_α^β of functions $\varphi(x)$ with properties (6) on the whole axis $(-\infty < x < \infty)$ is nonempty for $\alpha + \beta \geq 1$ $(\alpha, \beta \neq 0)$; moreover, for $\beta > 0$ the inequalities (6) on the whole axis x are equivalent to the following inequalities on the whole complex plane:

$$\varphi(x+iy) \leq C_\varphi \exp\left[-C_1|x|^{\frac{1}{\alpha}} + C_2|y|^{\frac{1}{1-\beta}}\right]. \tag{6'}$$

Since there exist entire functions of order p $(1/2 < p < 1)$ which decrease exponentially with order p on the half-axis $x \geq 0$ (and increase on the half-axis $x \leq 0$)* , the space $\pi\sigma_\alpha^\beta$ of basic functions $\varphi(x)$ with properties (6) on the half-axis $x \geq 0$, or, as is easy to verify, equivalently, with properties $(6')$ in the right half-plane, is nonempty even for $\beta > -1$ and $\alpha + \beta = 1$, where β may be equal to zero. In addition, as in (2), one can prove that $\pi\sigma_\alpha^\beta$ is a space sufficiently rich in functions**.

After these remarks one can show that the space Φ of entire functions with properties (6) and (7) is nonempty and sufficiently rich in functions. Indeed, first take the space $\pi\sigma_{\alpha'}^{\beta'}$ $(\beta' > -1,\ \alpha' + \beta' = 1)$ with elements $\psi(x)$. With the aid of the easily verified inequalities (3)

$$A_1|\xi|^h - A_2|\eta|^h \leq |\xi - \eta|^h \leq \max(2^{h-1}, 1)(|\xi|^h + |\eta|^h), \qquad h \geq 0,$$

and the known Young inequality

$$|\xi\eta| \le \frac{\varepsilon^r|\xi|^r}{r} + \frac{|\eta|^{r'}}{\varepsilon^{r'} r'}, \qquad \varepsilon > 0, \qquad r > 1, \qquad \frac{1}{r} + \frac{1}{r'} = 1,$$

noting that $\frac{1}{\alpha'} = \frac{1}{1-\beta'}$, one can prove that for any integer n, for any $\psi(x) \in \pi\sigma_{\alpha'}^{\beta'}$, we have

$$|\varphi(z)| = |\psi(z^n)| \le C_\psi \exp\left[-A_1|\operatorname{Re} z^n|^{\frac{1}{\alpha'}} + A_2|\operatorname{Im} z^n|^{\frac{1}{1-\beta'}}\right] \le$$

$$\le C_\varphi \exp\left[-B_1|x|^{\frac{1}{\alpha}} + B_2|x|^{\frac{1}{1-\beta}}\right],$$

where

$$\frac{1}{\alpha} = \frac{n}{\alpha'}, \qquad \frac{1}{1-\beta} = \frac{n}{1-\beta'} \quad (\beta' > -1).$$

Hence

$$\alpha = \frac{\alpha'}{n}, \qquad \beta = \frac{\beta'}{n} + \frac{n-1}{n}. \tag{8}$$

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* The existence of such entire functions is easily established by applying V. L. Bernstein' s theorem to the trigonometrically convex periodic function with period 2π,

$$h(\theta) = A\cos p(\pi - \theta),$$

as an indicator, where A is any positive number [(4)].

** That is, if for some function $f(x)$ and every $\varphi(x) \in \pi\sigma_\alpha^\beta$

$$\int_0^\infty f(x)\varphi(x)\,dx = 0,$$

then $f(x) \equiv 0$ almost everywhere on $(0, \infty)$.

Let us note that, since $n \geqslant 2$ and $\beta' > -1$, the inequality

$$\beta > \frac{n-2}{n} \geqslant 0. \tag{9}$$

holds.

Thus, if $\psi(x) \in \pi\sigma_{\alpha'}^{\beta'}$, then $\psi(x^n) \in \pi\sigma_\alpha^\beta$; therefore, the space $\pi\sigma_\alpha^\beta$ contains all functions of the form

$$\varphi(x) = x^{n-m-1}\varphi(x^n), \qquad \psi \in \pi\sigma_{\alpha'}^{\beta'}.$$

Since $\psi(x)$ are entire functions, we may expand them in series

$$\psi(x) = \sum_{k=0}^{\infty} a_k x_k.$$

It is easy to verify that all functions of the form

$$\varphi(x) = x^{n-m-1}\psi(x^n) = x^{n-m-1}\sum_{k=0}^{\infty} a_k x^{nk}$$

possess the properties (7). Hence the space Φ is nonempty.

Let us show that the space Φ is sufficiently rich in functions. Indeed, if for all $\varphi(x) \in \Phi$

$$\int_0^{\infty} f(x)\varphi(x)\,dx = 0,$$

where $f(x)$ is some function satisfying the inequality

$$|f(x)| \leqslant C_1 \exp\left[C_2|x|^{\frac{1}{\alpha}-\varepsilon}\right],$$

so that the integrals $\int_0^{\infty} f(x)\varphi(x)\,dx$ make sense for all $\varphi(x) \in \Phi$, then for any $\psi \in \pi\sigma_{\alpha'}^{\beta'}$ we have

$$\int_0^{\infty} f(x)x^{n-m-1}\psi(x^n)\,dx = \int_0^{\infty} \frac{f(\sqrt[n]{y})}{n\sqrt[n]{y^m}}\psi(y)\,dy = 0, \qquad y = x^n.$$

Since $m/n < 1$, $f(\sqrt[n]{y})/\sqrt[n]{y^m}$ is locally integrable in a neighborhood of the point $x = 0$. Since the space $\pi\sigma_{\alpha'}^{\beta'}$ is sufficiently rich in functions, it follows that

$$f(\sqrt[n]{y})/\sqrt[n]{y^m} = 0,$$

whence $f(x) \equiv 0$ almost everywhere on $(0, \infty)$.

We shall now solve in the basic space Φ the mixed problem adjoint to (1)–(3):

$$\frac{\partial\varphi}{\partial t} = -P^*(t, D^*)\varphi, \qquad D^* = (-1)^{nD}; \tag{1'}$$

$$\varphi(x, t_0) = \varphi_0(x), \qquad t_0 > 0 \text{ arbitrary}, \qquad \varphi_0 \in \Phi; \tag{2'}$$

$$\varphi(0) = \varphi'(0) = \cdots = \varphi^{(n-m-2)}(0) = \varphi^{(n-m)}(0) = \cdots = \varphi^{(n-1)}(0) = 0, \tag{3'}$$

where P^* is the matrix transposed to the matrix P.

Since, by the hypothesis of the theorem, system (1) has reduced order $p_0 < \frac{n}{n-2}$, and since system $(1')$ has the same reduced order p_0, the operator $Q^*(D^*, t_0, t)$ (^{1},^{2}) is bounded in the space Φ.

with $\beta = \frac{1}{p_0} > \frac{n-2}{n}$ (1), so that, according to (9) and (8), we have $\beta' > -1$, and therefore $\pi_{\alpha'}^{\beta'}$, and hence Φ, is nonempty.

As is easy to verify, the solution of the mixed problem $(1') - (3')$ is given by the formula

$$\varphi(x, t) = Q^*(D^*, t_0, t)\varphi_0(x). \tag{10}$$

Indeed, we have (2)

$$\frac{\partial \varphi}{\partial t} = -P^* Q^* \varphi_0 = -P^* \varphi, \qquad \varphi(x, t_0) = \varphi_0(x).$$

Finally, since Q^* is an entire function of the operator $\partial^n / \partial x^n$ and $\varphi_0(x)$ possesses the properties (7), it follows that $\varphi(x, t)$ also satisfies the conditions $(3')$.

Thus, the application of the operator Q^* to each $\varphi_0(x)$ from Φ gives a solution of the mixed problem $(1') - (3')$.

Now suppose that the mixed problem (1)–(3) has a solution $u(x, t)$ satisfying condition (5). We shall prove that it is the unique solution of the given problem. Indeed, by virtue of (5), $u(x, t)$ is a functional in the space Φ, and, under conditions (3) and properties (7), for all $\varphi_0 \in \Phi$ we have

$$(Pu, \varphi_0) = (u, P^* \varphi_0), \qquad (Pu, Q^* \varphi_0) = (u, P^* Q^* \varphi_0).$$

Therefore, for any $\varphi_0 \in \Phi$,

$$\frac{\partial}{\partial t}(u, Q^* \varphi_0) = (u_t, Q^* \varphi_0) + \left(u, \frac{\partial Q^*}{\partial t} \varphi_0\right) = (Pu, Q^* \varphi_0) + (u, -P^* Q^* \varphi_0) =$$
$$= (u, P^* Q^* \varphi_0) - (u, P^* Q^* \varphi_0) = 0.$$

If for $t = 0$ $u(x, 0) = 0$, then for all $\varphi_0 \in \Phi$, for every $t \geq 0$,

$$(u, Q^* \varphi_0) = 0.$$

In particular, for $t = t_0$ we have, for all $\varphi_0 \in \Phi$,

$$(u, \varphi_0) = 0.$$

Hence, by virtue of the sufficient supply of functions in the space Φ and the arbitrariness of $t_0 > 0$, we obtain

$$u(x, t) \equiv 0,$$

as was required to prove.

Clearly, the theorem is also valid in the multidimensional case $x = (x_1, \ldots, x_L)$ in domains where all or some of the $x_j \geq 0$.

In conclusion I take the opportunity to express my deep gratitude to G. E. Shilov for valuable advice in carrying out this work.

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[3] G. E. Shilov, *Dokl. Akad. Nauk SSSR*, **102**, 5 (1955).
[4] B. Ya. Levin, *Distribution of Zeros of Entire Functions*, Moscow, 1956, p. 124.

Note: Figure translations are in progress. See original paper for figures.