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**Abstract**

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# THE FIRST BOUNDARY VALUE PROBLEM FOR AN EQUATION OF ELLIPTIC TYPE OF FOURTH ORDER DEGENERATING ON THE BOUNDARY OF THE DOMAIN

*(Presented by Academician S. L. Sobolev on 16 XI 1956)*

In the present note the first boundary value problem is considered for the general fourth-order equation

$$
\begin{aligned}
Lu \equiv -\Bigg\{ & \frac{\partial^2}{\partial x^2}\left(a_{1111}\frac{\partial^2 u}{\partial x^2}\right) + \frac{\partial^2}{\partial x \partial y}\left(a_{1212}\frac{\partial^2 u}{\partial x \partial y}\right) + \frac{\partial^2}{\partial y^2}\left(a_{2222}\frac{\partial^2 u}{\partial y^2}\right) \\
& + \frac{1}{2}\frac{\partial^2}{\partial x \partial y}\left(a_{1112}\frac{\partial^2 u}{\partial x^2}\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(a_{1112}\frac{\partial^2 u}{\partial x \partial y}\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(a_{1222}\frac{\partial^2 u}{\partial x \partial y}\right) \\
& + \frac{1}{2}\frac{\partial^2}{\partial x \partial y}\left(a_{1222}\frac{\partial^2 u}{\partial y^2}\right)\Bigg\} + a_{111}\frac{\partial^3 u}{\partial x^3} + a_{122}\frac{\partial^3 u}{\partial x \partial y^2} + a_{112}\frac{\partial^3 u}{\partial x^2 \partial y} \\
& + a_{222}\frac{\partial^3 u}{\partial y^3} + a_{11}\frac{\partial^2 u}{\partial x^2} + 2a_{12}\frac{\partial^2 u}{\partial x \partial y} + a_{22}\frac{\partial^2 u}{\partial y^2} \\
& + a_1\frac{\partial u}{\partial x} + a_2\frac{\partial u}{\partial y} + a_0 u = h
\end{aligned}
\tag{1}
$$

of elliptic type with two independent variables, degenerating on a part of the boundary of the domain adjacent to the axis $Ox$. Equation (1) is considered in a finite domain $D$, situated in the upper half-plane and having a part of its boundary $\Gamma_0$ on the axis $Ox$. The remaining part of the boundary we denote by $\Gamma_1$. We assume that the boundary $\Gamma_1$ is such that the embedding theorems of S. L. Sobolev ([1]) hold for it.

We suppose that the coefficients of the highest derivatives in equation (1) are continuous in the closed domain $\overline{D} \equiv D \cup \Gamma$, $\Gamma = \Gamma_1 \cup \Gamma_0$, and twice continuously differentiable in $D^\delta = D \cap (y > \delta)$, where $\delta > 0$ is arbitrary; $a_{111}$, $a_{112}$, $a_{122}$, and $a_{222}$ are three times continuously differentiable in $D^\delta$; $a_{11}$, $a_{12}$, and $a_{22}$ are twice continuously differentiable in $D^\delta$; $a_1$ and $a_2$ are continuously differentiable in $D^\delta$; $a_0$ is continuous in $D^\delta$. We also assume that

$$c^2 y^\alpha \leq a_{2222} \leq C^2 y^\alpha,$$

where $\alpha \geq 0$. If $\alpha = 0$, then we shall assume that $a_{1111} \to 0$ as $y \to 0$. For definiteness we shall suppose that at every point of the domain $\overline{D}$ the form

$$A^4(\xi_1, \xi_2; x, y) \equiv a_{1111}\xi_1^4 + a_{1212}\xi_1^2\xi_2^2 + a_{2222}\xi_2^4 + a_{1112}\xi_1^3\xi_2 + a_{1222}\xi_1\xi_2^3 \geq 0$$

for $\xi_1^2 + \xi_2^2 > 0$, with equality attained only at the points of $\Gamma_0$.

**Theorem 1.** Let, for any real numbers $\xi_1, \xi_2$ $(\xi_1^2 + \xi_2^2 > 0)$, the form $A^4(\xi_1, \xi_2; x, y) \geq 0$, where equality is attained only at the points $\Gamma_0$. Make the substitution: $\xi_1^2 = \xi_{11}$, $\xi_1\xi_2 = \xi_{12}$, $\xi_2^2 = \xi_{22}$. Then there exists a sufficiently smooth function $\lambda = \lambda(x, y)$ such that the form

$$B(\xi_{11}, \xi_{12}, \xi_{22}; x, y) \equiv A^2(\xi_{11}, \xi_{12}, \xi_{22}; x, y) + \lambda(\xi_{12}^2 - \xi_{11}\xi_{12}) \equiv$$

$$\equiv a_{1111}\xi_{11}^2 + (a_{1212} + \lambda)\xi_{12}^2 + a_{2222}\xi_{22}^2 + a_{1112}\xi_{11}\xi_{12} - a_{1222}\xi_{12}\xi_{22} - \lambda\xi_{11}\xi_{22} \geq 0$$

provided that $\xi_{11}^2 + \xi_{12}^2 + \xi_{22}^2 > 0$; $\xi_{11}, \xi_{12}, \xi_{22}$ are arbitrary real numbers, and the equality sign can occur only at points of $\Gamma_0$.

As in the paper ([3]), the space $\dot{\Omega}$ and the space $\dot{R}$ are introduced, with metric defined by the scalar product

$$\begin{aligned}\{Gu, Gv\} = \iint_D \Bigg[ & a_{1111}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + (a_{1212} + \lambda)\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial^2 v}{\partial x\,\partial y} + a_{2222}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} \\ & + \frac{1}{2}a_{1112}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x\,\partial y} + \frac{1}{2}a_{1112}\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}a_{1222}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x\,\partial y} \\ & + \frac{1}{2}a_{1222}\frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial^2 v}{\partial y^2} + \frac{\lambda}{2}\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} + \frac{\lambda}{2}\frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}\Bigg] dx\,dy,\end{aligned} \tag{2}$$

which is possible by virtue of Theorem 1.

We assume that, for $\alpha \geq 0$, $\beta \geq 0$, the following conditions are satisfied:

$$c_1^2 y^\beta \leq a_{1212} + \lambda \leq C_1^2 y^\beta, \qquad 0 \leq y^\alpha \xi_{22}^2 \leq C_2^2 B(\xi_{11}, \xi_{12}, \xi_{22}; x, y),$$

$$0 \leq y^\beta \xi_{12}^2 \leq C_3^2 B(\xi_{11}, \xi_{12}, \xi_{22}; x, y), \qquad 0 \leq a_{1111}\xi_{11}^2 \leq C_4^2 B(\xi_{11}, \xi_{12}, \xi_{22}; x, y)$$

for arbitrary real $\xi_{11}, \xi_{12}, \xi_{22}$.

We introduce the notation:

$$N_1 = a_2 - 2\frac{\partial \bar{a}_{12}}{\partial x} - \frac{\partial \bar{a}_{22}}{\partial y} + \frac{\partial^2 a_{122}}{\partial x\,\partial y} + \frac{\partial^2 a_{222}}{\partial y^2}, \qquad N_2 = 2\frac{\partial a_{222}}{\partial y} + \frac{\partial a_{122}}{\partial x} - \bar{a}_{22},$$

$$\iint_D uv\,dx\,dy = [u, v],$$

$$\bar{a}_{11} = a_{11} - \frac{1}{2}\frac{\partial^2 \lambda}{\partial y^2}, \qquad \bar{a}_{12} = a_{12} + \frac{1}{2}\frac{\partial^2 \lambda}{\partial x\,\partial y}, \qquad \bar{a}_{22} = a_{22} - \frac{1}{2}\frac{\partial^2 \lambda}{\partial x^2}.$$

By $\mathcal{L}^2_D(\sigma)$, $\sigma = \sigma(x, y)$, $(x, y) \in D$, we shall denote the space of functions square-summable with weight $\sigma(x, y)$ over the domain $D$: $[\sigma u, u] < +\infty$.

**Case a)** For $0 \le \alpha < 1$, $\beta \ge 0$, the first boundary-value problem for equation (1) with homogeneous boundary conditions is posed as follows: find a generalized solution of equation (1) which vanishes on $\Gamma = \Gamma_1 \cup \Gamma_0$ together with its normal derivative.

Suppose that in some neighborhood of $\Gamma_0$ the following conditions are satisfied:

**Case b)** $1 \le \alpha < 3$, $\beta \ge 1$; $0 \le \beta < 1$, $\alpha \ge 1$:

$$\begin{aligned} 1)\quad N_1 &\ge -c_1^2 y^{-2}|\ln y|^{-1}(\ln|\ln y|)^{-\varepsilon_1} && \text{for } \alpha = 1,\\ N_1 &\ge -c_1^2 y^{\alpha-3}(\ln|\ln y|)^{-\varepsilon_1} && \text{for } 1 < \alpha < 2,\\ N_1 \ge -c_1^2 y^{\beta-1} \text{ or } N_1 &\ge -c_1^2 y^{\alpha-3}(\ln|\ln y|)^{-\varepsilon_1} && \text{for } 2 \le \alpha < 3,\\ N_1 &\ge -c_1^2 y^{\beta-1} && \text{for } \alpha \ge 3; \end{aligned}$$

$$\begin{aligned} 2)\quad N_2 &\le c_2^2 y^{-1}|\ln y|^{-1}(\ln|\ln y|)^{-\varepsilon_1} && \text{for } \alpha = 1,\\ N_2 &\le c_2^2 y^{\alpha-2}(\ln|\ln y|)^{-\varepsilon_1} && \text{for } 1 < \alpha < 2,\\ N_2 \le c_2^2 y^{\beta} \text{ or } N_2 &\le c_2^2 y^{\alpha-2}(\ln|\ln y|)^{-\varepsilon_1} && \text{for } 2 \le \alpha < 3,\\ N_2 &\le c_2^2 y^{\beta} && \text{for } \alpha \ge 3; \end{aligned}$$

$$\begin{aligned} 3)\quad \text{either}\quad |a_{222}| &\le c^2|\ln y|^{-1}(\ln|\ln y|)^{-\varepsilon_1} && \text{for } \alpha = 1,\\ |a_{222}| &\le c^2 y^{\alpha-1}(\ln|\ln y|)^{-\varepsilon_1} && \text{for } 1 < \alpha < 3,\ \alpha > 3,\\ |a_{222}| &\le c^2 y^2|\ln y|^{-1}(\ln|\ln y|)^{-\varepsilon_1} && \text{for } \alpha = 3; \end{aligned} \tag{3}$$

$$\begin{aligned} |a_{122}| &\le c_3^2|\ln y|^{-1}(\ln|\ln y|)^{-\varepsilon_1} && \text{for } \alpha = 1,\ \beta \le 1;\ \alpha \le 1,\ \beta = 1,\\ |a_{122}| &\le c_3^2 y^{\gamma-1}(\ln|\ln y|)^{-\varepsilon_1} && \text{for } \alpha \ge 1,\ \beta > 1;\ \alpha > 1,\ \beta \ge 1; \end{aligned} \tag{4}$$

$$
\begin{aligned}
a_{112} &\le c_4^2 y^{\beta-1} \quad \text{for } 0 \le \beta < 1, \\
a_{122} &\le c_4^2 |\ln y|^{-1} (\ln|\ln y|)^{-\varepsilon_1} \quad \text{for } \beta = 1, \\
a_{112} &\le c_4^2 y^{\beta-1} (\ln|\ln y|)^{-\varepsilon_1} \quad \text{for } \beta > 1;
\end{aligned}
\tag{5}
$$

or

$$
\begin{gathered}
-\tilde{c}^2 |\ln y|^{-1} (\ln|\ln y|)^{-\varepsilon_1} \le a_{222} \le 0 \quad \text{for } \alpha = 1, \\
-\tilde{c}^2 y^{\alpha-1} (\ln|\ln y|)^{-\varepsilon_1} \le a_{222} \le 0 \quad \text{for } 1 < \alpha < 3,\ \alpha > 3, \\
-\tilde{c}^2 y^2 |\ln y|^{-1} (\ln|\ln y|)^{-\varepsilon_1} \le a_{222} \le 0 \quad \text{for } \alpha = 3; \\
a_{112} \le 0; \qquad a_{122}^2 - 3a_{112}a_{222} \le 0.
\end{gathered}
\tag{6}
$$

Case c) $\alpha \ge 3,\ \beta \ge 1$:

1)
$$
\begin{gathered}
N_1 \ge -c_1^2 |\ln y|^{-1} (\ln|\ln y|)^{-\varepsilon_1} \quad \text{for } \alpha = 3,\ \beta \ge 1;\ \alpha \ge 3,\ \beta = 1, \\
N_1 \ge -c_1^2 y^{\alpha-3} (\ln|\ln y|)^{-\varepsilon_1} \quad \text{or} \quad N_1 \ge -c_1^2 y^{\beta-1} (\ln|\ln y|)^{-\varepsilon_1} \\
\text{for } \alpha > 3,\ \beta > 1;
\end{gathered}
$$

2)
$$
\begin{gathered}
N_2 \le c_2^2 y |\ln y|^{-1} (\ln|\ln y|)^{-\varepsilon_1} \quad \text{for } \alpha \ge 3,\ \beta = 1;\ \alpha = 3,\ \beta \ge 1, \\
N_2 \le c_2^2 y^{\alpha-2} (\ln|\ln y|)^{-\varepsilon_1} \quad \text{or} \quad N_2 \le c_2^2 y^{\beta} (\ln|\ln y|)^{-\varepsilon_1} \\
\text{for } \alpha > 3,\ \beta > 1;
\end{gathered}
$$

3) either (3), (4), and (5) for $\alpha \ge 3,\ \beta \ge 1$, or (6) for $\alpha \ge 3$, where $\varepsilon_1$ is an arbitrary positive number, $c$ with subscripted values are certain constants, $\gamma = \max(\alpha, \beta)$.

The conditions just listed will be called "conditions $S$."

When conditions $S$ are satisfied, the first boundary-value problem with homogeneous boundary conditions consists in finding a generalized solution of equation (1): in case b), one that vanishes on $\Gamma$, whose first derivatives vanish on $\Gamma_1$; on $\Gamma_0$ no conditions are imposed on the first derivatives of the sought solution; in case c), one that vanishes on $\Gamma_1$ together with its first derivatives; on $\Gamma_0$ no boundary conditions at all are imposed.

We note that the required boundary conditions are satisfied, on the basis of the embedding theorems proved in note ([3]), by functions from $\dot{\Omega}$. The vanishing of a function and of derivatives on the boundary is understood in the mean.

Under the conditions formulated above, by a generalized solution of the first boundary-value problem for equation (1) in cases a), b), and c) we mean such a function $u \in \dot{\Omega}$ that, for any function $v \in \dot{\Omega}$ that vanishes in some neighborhood of $\Gamma_0$, the following integral relation is fulfilled:

$$
\begin{aligned}
[h, v] = &- \{Gu, Gv\} + \left[\frac{\partial u}{\partial x},\, a_{111}\frac{\partial^2 v}{\partial x^2} + a_{112}\frac{\partial^2 v}{\partial x\,\partial y}\right] \\
&+ \left[\frac{\partial u}{\partial x},\, \left(2\frac{\partial a_{111}}{\partial x} + \frac{\partial a_{112}}{\partial y} - \bar{a}_{11}\right)\frac{\partial v}{\partial x} + \left(\frac{\partial a_{112}}{\partial x} - \bar{a}_{12}\right)\frac{\partial v}{\partial y}\right] \\
&+ \left[\frac{\partial u}{\partial y},\, \left(2\frac{\partial a_{222}}{\partial y} + \frac{\partial a_{122}}{\partial x} - \bar{a}_{22}\right)\frac{\partial v}{\partial y} + \left(\frac{\partial a_{122}}{\partial y} - \bar{a}_{12}\right)\frac{\partial v}{\partial x}\right] \\
&+ \left[\frac{\partial u}{\partial x},\, \left(\frac{\partial^2 a_{111}}{\partial x^2} + \frac{\partial^2 a_{112}}{\partial x\,\partial y} - \frac{\partial a_{11}}{\partial x} - \frac{\partial a_{12}}{\partial y}\right) v\right] \qquad (7) \\
&+ \left[\frac{\partial u}{\partial y},\, a_{222}\frac{\partial^2 v}{\partial y^2} + a_{122}\frac{\partial^2 v}{\partial x\,\partial y}\right] \\
&+ \left[\frac{\partial u}{\partial y},\, \left(\frac{\partial^2 a_{222}}{\partial y^2} + \frac{\partial^2 a_{122}}{\partial x\,\partial y} - \frac{\partial a_{22}}{\partial y} - \frac{\partial a_{12}}{\partial x}\right) v\right] \\
&+ \left[u,\, \left(a_0 - \frac{\partial a_1}{\partial x} - \frac{\partial a_2}{\partial y}\right) v\right] - \left[u,\, a_1\frac{\partial v}{\partial x} + a_2\frac{\partial v}{\partial y}\right].
\end{aligned}
$$

**Theorem 2.** Suppose that in cases b) and c) the conditions $S$ are satisfied. Suppose also that in case a), in some neighborhood of $\Gamma_0$, the inequalities hold:

$$N_1 \geqslant c_1^2 y^{\alpha-3}; \qquad N_2 \leqslant c_2^2 y^{\alpha-2};$$

or (5) and

$$|a_{222}| \leqslant c^2 y^{\alpha-1},$$

$$|a_{122}| \leqslant c_3^2 y^{\gamma-1}(|\ln|(\ln y)||)^{-\varepsilon_1} \qquad \text{for } 0 \leqslant \beta < 1,$$

$$|a_{122}| \leqslant c_3^2 y^{\beta-1}(\ln|\ln y|)^{-\varepsilon_1} \qquad \text{for } \beta > 1,$$

$$|a_{122}| \leqslant c_3^2 |\ln y|^{-1}(\ln|\ln y|)^{-\varepsilon_1} \qquad \text{for } \beta = 1;$$

or

$$-\tilde{c}^2 y^{\alpha-1} \leqslant a_{222} \leqslant 0, \quad a_{112} \leqslant 0, \quad a_{122}^2 - 3a_{112}a_{222} \leqslant 0,$$

where $\gamma = \max(\alpha, \beta)$, $\varepsilon_1 > 0$ and is arbitrary. If, in addition to the conditions listed above, in each case, for any point of the domain $D$, the following is satisfied:

$$a_0 - \frac{1}{2}\frac{\partial a_1}{\partial x} - \frac{1}{2}\frac{\partial a_2}{\partial y} + \frac{1}{2}\frac{\partial^2 a_{11}}{\partial x^2} + \frac{\partial^2 a_{12}}{\partial x\,\partial y} + \frac{1}{2}\frac{\partial^2 a_{22}}{\partial y^2} -$$

$$-\frac{1}{2}\frac{\partial^3 a_{111}}{\partial x^3} - \frac{1}{2}\frac{\partial^3 a_{112}}{\partial x^2 \partial y} - \frac{1}{2}\frac{\partial^3 a_{122}}{\partial x\,\partial y^2} - \frac{1}{2}\frac{\partial^3 a_{222}}{\partial y^3} \leqslant 0;$$

$$\frac{3}{2}\frac{\partial a_{111}}{\partial x} + \frac{1}{2}\frac{\partial a_{112}}{\partial y} - \bar{a}_{11} \leqslant 0, \qquad \frac{3}{2}\frac{\partial a_{222}}{\partial y} + \frac{1}{2}\frac{\partial a_{122}}{\partial x} - \bar{a}_{22} \leqslant 0,$$

$$\left(\frac{3}{2}\frac{\partial a_{111}}{\partial x} + \frac{1}{2}\frac{\partial a_{112}}{\partial y} - \bar{a}_{11}\right)\left(\frac{3}{2}\frac{\partial a_{222}}{\partial y} + \frac{1}{2}\frac{\partial a_{122}}{\partial x} - \bar{a}_{22}\right) -$$

$$-\left(\frac{1}{2}\frac{\partial a_{112}}{\partial x} + \frac{1}{2}\frac{\partial a_{122}}{\partial y} - \bar{a}_{12}\right)^2 \geqslant 0,$$

then the first boundary-value problem in the formulation indicated above has a unique solution for any right-hand side $h \in \mathcal{L}_2^2(\sigma_0^{-1})$, where $\sigma_0$ is the weight function defined in the embedding theorems ([3]).

The proof is carried out by the functional method, the same as in the work of M. I. Vishik ([2]). The lower-order terms of equation (1) are realized in the form of an operator acting in $\mathring{R}$. It is proved that the operator corresponding to the operator $\bar{L}$ has everywhere a dense domain of definition and a bounded inverse operator.

In conclusion I take this opportunity to express my sincere gratitude to Academician S. L. Sobolev, under whose supervision the present work was carried out.

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*Note: Figure translations are in progress. See original paper for figures.*