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Abstract

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ON A QUASICLASSICAL INTERPRETATION OF QUANTUM EFFECTS IN THE THEORY OF A RADIATING ELECTRON

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1. In the present work we wish to give a quasiclassical interpretation of the quantum effects which, according to the conclusions of quantum theory, should manifest themselves in the motion of an ultrarelativistic electron in a magnetic field ([1,2]).

As is known, the classical equations describing the motion of an ultrarelativistic electron in a synchrotron, with radiation taken into account, have the form

$$\frac{E\ddot{z}}{c^2} + \frac{\dot{E}\dot{z}}{c^2} - \frac{er\dot{\varphi}H_r}{c} = -\frac{\dot{z}W^{\mathrm{cl}}}{c^2}, \tag{1}$$

$$\frac{E\ddot{r}}{c^2} + \frac{\dot{E}\dot{r}}{c^2} - \frac{Er\dot{\varphi}^2}{c^2} + \frac{er\dot{\varphi}H_z}{c} = -\frac{\dot{r}W^{\mathrm{cl}}}{c^2}, \tag{2}$$

$$\dot{E} = \dot{E}^{\mathrm{ext}} - W^{\mathrm{cl}}, \tag{3}$$

where r, z, φ are cylindrical coordinates; $H_z = Br^{-q}$, $H_r = -qzBr^{-(q+1)}$ are the components of the magnetic-field intensity; q is the index of decrease of the magnetic field ($0 < q < 1$). In the classical case, for the radiation energy we shall have the expression

$$W^{\mathrm{cl}} = \frac{R^2}{\rho^2}\overline{W}^{\mathrm{cl}}, \qquad \overline{W}^{\mathrm{cl}} = \frac{2}{3}\frac{e^2c}{R^2}\left(\frac{E}{m_0c^2}\right)^4, \tag{4}$$

where $\overline{W}^{\mathrm{cl}}$ is the mean radiation energy, which does not depend on the coordinate $x = r - R$ of the betatron oscillations, while the radius of the instantaneous equilibrium orbit R and the radius of curvature ρ may be found from the equality

$$E = eBR^{1-q} = eBr^{-q}\rho. \tag{5}$$

Hence, in the linear approximation we find

$$\rho \simeq R + qx. \tag{6}$$

The expression for the energy received by the electron from the accelerating device (the fundamental harmonic), in the linear approximation of phase oscillations, may be represented in the form

$$\dot{E}^{\text{ext}} = \frac{cV_0}{2\pi r}\sin(\varphi - \omega_0 t) = \frac{cV_0}{2\pi r}(\sin\varphi_0 + \psi\cos\varphi_0), \tag{7}$$

where V_0/e is the amplitude of the potential in the accelerating gap; ω_0 is the angular frequency of the high-frequency field, related to the radius of the synchrotron equilibrium orbit by the relation $R_0 = c/\omega_0$. Here $\omega_0 t + \varphi_0$ characterizes the uniform rotation of the electron, while $\psi = \varphi - \omega_0 t - \varphi_0$ contains only the oscillatory part.

Taking the latter relations into account, instead of (1)–(3) we shall have (in the linear approximation):

$$\ddot{z} + \gamma\dot{z} + \omega_z^2 z = -\frac{\overline{W}^{\text{cl}}}{E}\dot{z}; \tag{8}$$

$$\ddot{x} + \gamma\dot{x} + \omega_r^2 x = -\frac{\overline{W}^{\text{cl}}}{E}\dot{x} - \ddot{R} - \frac{\dot{x}\dot{R}}{R}, \tag{9}$$

$$\dot{r}E = (1-q)E\left(\dot{R} + \frac{x\dot{R}}{R}\right) = \frac{V_0 c}{2\pi}(\sin\varphi_0 + \psi\cos\varphi_0) - r\overline{W}^{\text{cl}}, \tag{10}$$

where $\omega_z = \sqrt{q}\,c/R, \quad \omega_r = \sqrt{1-q}\,c/R$. Eliminating from the right-hand side of (9) the quantity $\ddot{R} + \dot{x}\dot{R}/R$, we find, for the description of radial betatron oscillations, the equation

$$\ddot{x} + \gamma\dot{x} + \omega_r^2 x = -\frac{q}{1-q}\frac{\overline{W}^{\text{cl}}}{E}\dot{x}, \tag{11}$$

i.e., upon introducing a continuous friction force in the theory of betatron oscillations, along with the "small" damping coefficient $\gamma = \dot{E}/E$, one must also take into account the large damping coefficients ([3–5])

$$\Gamma_z = \frac{\overline{W}^{\text{cl}}}{E} \ll \gamma, \qquad \Gamma_r = \frac{q}{1-q}\Gamma_z. \tag{12}$$

Introducing the coordinate $X = R - R_0$, related to the phase ψ by means of the relation $X = -R_0^2\dot{\psi}/c$, and taking into account that $\overline{W}^{\mathrm{cl}} = (R/R_0)^{2-4q}\, W_0$ (W_0 corresponds to $R = R_0$), we obtain the following equation for determining the phase oscillations:

$$\ddot{\psi} + \Gamma\dot{\psi} + \Omega^2\psi = -\frac{c}{(1-q)R_0^2 E_0}\left(\frac{cV_0 \sin\varphi_0}{2\pi} - R_0 W_0\right) = 0, \tag{13}$$

where the point of equilibrium phase must be found from the equation

$$\frac{cV_0}{2\pi R_0}\sin\varphi_0 = W_0; \tag{14}$$

the angular frequency of the slow synchrotron oscillations is equal to

$$\Omega = \sqrt{\frac{c^2 V_0 \cos\varphi_0}{2\pi(1-q)R_0^2 E_0}} = \sqrt{\frac{cW_0 \operatorname{ctg}\varphi_0}{(1-q)R_0 E_0}} \ll \omega_r; \tag{15}$$

$$\Gamma = \frac{3-4q}{1-q}\frac{W_0}{E} \sim \Gamma_r$$

is the "large" damping coefficient of synchrotron oscillations.

2. As is known, in the radiation of a luminous electron it is necessary to estimate the influence of the discreteness factor of the radiation, since, according to quantum theory, during one revolution a comparatively small number of quanta is emitted, $N \sim \alpha\, E/(m_0c^2)$ ([2]), especially when $E < E_{1/5}$.

To describe the discrete character of the radiation, we introduce the fluctuation force

$$\mathbf{F}^{\mathrm{fluct}} = -\frac{\mathbf{v}}{c^2}\sum_i \left\{\frac{R}{\rho}\,\varepsilon\,\delta(t-t_i) - \frac{R^2}{\rho^2}\overline{W}^{\mathrm{cl}}\right\}. \tag{16}$$

The introduction of the fluctuation force is formally equivalent to the fact that in equations (8), (9), (10) one should replace the quantities according to the scheme (**v**—no-

number of the emitted harmonic):

$$\overline{W}^{\mathrm{cl}} \to \sum_i \overline{\varepsilon}\delta(t-t_i) = \sum_i \hbar\frac{c}{R}\gamma\delta(t-t_i),$$

$$W^{\mathrm{cl}} \to \frac{R}{\rho}\sum_i \overline{\varepsilon}\delta(t-t_i) = \sum_i \left(1 - q\frac{x}{R}\right)\hbar\frac{c}{R}\gamma\delta(t-t_i). \tag{17}$$

Then these equations take the form

$$\ddot{z}+\gamma\dot{z}+\omega_z^2 z=-\sum_i \frac{\dot{z}}{E}\,\overline{\varepsilon}\delta(t-t_i)=-\sum_i \frac{c\cos\theta}{E}\,\overline{\varepsilon}\delta(t-t_i); \tag{18}$$

$$\ddot{x}+\gamma\dot{x}+\omega_r^2 x=-\sum_i \frac{\dot{x}}{E}\,\overline{\varepsilon}\delta(t-t_i)-\ddot{R}-\frac{\dot{x}\dot{R}}{R}, \tag{19}$$

$$r\dot{E}=(1-q)E\left(\dot{R}+\frac{x\dot{R}}{R}\right)=\frac{cV_0}{2\pi}(\sin\varphi_0+\psi\cos\varphi_0)-\sum_i R\overline{\varepsilon}c\left[1+(1-q)\frac{x}{R}\right]\delta(t-t_i). \tag{20}$$

Equation (18) characterizes axial oscillations with allowance for quantum fluctuations.

In order to obtain the final equations for radial oscillations, we must substitute into the right-hand side of (19), instead of $\ddot{R}+\frac{x\dot{R}}{R}$, the value from (20), while discarding terms proportional to ψ. Then we find:

$$\ddot{x}+\gamma\dot{x}+\omega_r^2 x=\frac{R}{E(1-q)}\frac{d}{dt}\sum_i \overline{\varepsilon}\delta(t-t_i). \tag{21}$$

Let us note that the energy of the fluctuational radiation is inversely proportional to ρ (6), and not to ρ^2, as in the classical case. However, it should be taken into account that the transition probability is also inversely proportional to ρ:

$$w(\gamma,t_i)=\frac{R}{\rho}\,\overline{w}(\gamma,t_i), \tag{22}$$

where

$$\overline{w}(\gamma,t_i)=\frac{e^2}{\pi\sqrt{3}\,c\hbar}\frac{c}{R}\left(\frac{m_0c^2}{E}\right)^2\int_{\frac{2}{3}\gamma(m_0c^2/E)^3}^{\infty}K_{5/3}(\xi)\,d\xi. \tag{23}$$

Therefore, in passing to continuous radiation we again obtain the classical formula

$$\lim_{\Delta t_i\to 0}\sum_i \overline{\varepsilon}\delta(t-t_i)=\int_0^\infty d\gamma\int_0^\infty \frac{R}{\rho}\,\overline{\varepsilon}\,\frac{R}{\rho}\,\overline{w}(\gamma,t_i)\delta(t-t_i)\,dt_i=\left(\frac{R}{\rho}\right)^2\overline{W}^{\text{cl}}. \tag{24}$$

In exactly the same way, the "large" damping coefficients can be obtained by the limiting transition from quantum fluctuational forces to continuous radiation:

$$\lim_{\Delta t_i \to 0} \frac{R}{E(1-q)} \frac{d}{dt} \sum_i \overline{\varepsilon} \delta(t - t_i) =$$

$$= \frac{R}{E(1-q)} \frac{d}{dt} \int_0^\infty d\gamma \int_0^\infty \overline{\varepsilon} \left(1 - \frac{qx}{R}\right) \overline{w} \delta(t - t_i) \, dt_i = -\Gamma_r \dot{x} \qquad (25)$$

and so on.

For the square of the amplitude of radial oscillations, taking into account the fluctuation force, from (21) we find the value

$$a^2 = a_0^2 \exp\left[-\int_0^t \gamma(t')\, dt'\right] +$$

$$+ \frac{55}{24\sqrt{3}(1-q)^2} \frac{e^2}{m_0 c} \frac{\hbar}{m_0 c R_0} \int_0^t \exp\left[-\int_{t'}^t \gamma(t'')\, dt''\right] \left(\frac{E(t')}{m_0 c^2}\right)^5 dt'. \qquad (26)$$

To obtain the formula given in (2,6), one should take into account that

$$\exp\left[-\int_{t'}^t \gamma(t'')\, dt''\right] = \frac{E(t')}{E(t)}.$$

For axial oscillations the corresponding expression will be approximately $(E/m_0c^2)^2$ times smaller. In the same papers (3-5), it is proposed, along with the introduction of the fluctuation force acting on betatron oscillations, also to retain the "large" classical value for the damping coefficients (Γ_z, Γ_r).

Let us next examine synchrotron oscillations in the presence of quantum fluctuations. Setting in formula (20) $x = 0$ and $X = -\frac{R_0^2}{c}\dot{\psi}$, we find

$$R\dot{E} = -\frac{E_0 R_0^2 (1-q)}{c} \ddot{\psi} = \frac{E_0 R_0^2 (1-q) \Omega^2}{c} \psi + \frac{V_0 c \sin\varphi_0}{2\pi} - \sum_i \hbar\nu c \, \delta(t - t_i). \quad (27)$$

Taking into account that the period of synchrotron oscillations $T = 2\pi/\Omega \gg 2\pi/\omega_r$ is much greater than the mean time interval $\Delta t_i = t_{i+1} - t_i$ between two successive acts of radiation, we can average expression (27) over a time interval Δt satisfying the condition $\Delta t_i \ll \Delta t \ll T$. Then for the phase oscillations we obtain the classical equation, with the fluctuation quantum force turning into a friction force with a "large" damping coefficient Γ*. This is apparently connected with the circumstance that phase oscillations are caused by a change in the total energy, on which quantum effects should have an influence only at very large values $E \sim E_{l_2}$ (1), whereas the action of the fluctuation force on radial betatron oscillations should manifest itself at comparatively low energies $E \sim E_{l_s}$ (2).

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CITED LITERATURE

1. A. A. Sokolov, N. P. Klepikov, I. M. Ternov, ZhETF, **24**, 249 (1953).
2. A. A. Sokolov, I. M. Ternov, ZhETF, **25**, 698 (1953); DAN, **97**, 823 (1954).
3. A. A. Kolomensky, A. N. Lebedev, CERN, Symposium, **1**, Geneva, 1956, p. 447.
4. M. S. Livingston, CERN, Symposium, **1**, Geneva, 1956, p. 439.
5. I. Henry, Phys. Rev., **106**, 1057 (1957).
6. A. N. Matveev, DAN, **107**, 671 (1956).
7. M. Sands, Phys. Rev., **97**, 470 (1955).
8. A. N. Matveev, Nuovo Cim., **5**, 1782 (1957).

* In papers (3, 7, 8), devoted to phase oscillations, both the fluctuation force and the "large" damping coefficient are taken into account.

Note: Figure translations are in progress. See original paper for figures.