ON POLYNOMIALS ORTHOGONAL ON A SMOOTH CONTOUR WITH A DIFFERENTIABLE WEIGHT

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Abstract

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ON POLYNOMIALS ORTHOGONAL ON A SMOOTH CONTOUR WITH A DIFFERENTIABLE WEIGHT

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We consider the dependence of the polynomials $\{P_k(z)\}$, orthonormal with weight $n(z)$ on the boundary Γ of a certain bounded domain G, on the differential properties of the weight function and on the degree of smoothness of the boundary of the domain; generalized Faber polynomials are also considered. Denote by G_∞ the complement of $\overline{G}$, and let the function $w = \Phi(z)$ map the domain G_∞ onto the complement of the unit disk under the conditions $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$, while $z = \Psi(w)$ is the inverse function. We shall assume that the positive function $n(z)$ is differentiable on Γ p times and that the p-th derivative satisfies a Lipschitz condition of order $\alpha < 1$; Γ is a smooth curve, and, in addition, we shall characterize the degree of smoothness by the boundary properties of the mapping function $z = \Psi(w)$, namely, we shall assume that $\Psi^{(p+4)}(w) \in H_1$, where H_1 is the class of analytic functions representable by a Cauchy integral in terms of their angular boundary values.

As in the work ([3]), with the aid of Szegő's theorem we construct a function $D(w)$, analytic outside the unit disk, which is continuously differentiable in $|w| \geq 1$ p times, whose p-th derivative satisfies a Lipschitz condition of order α, and whose boundary values satisfy the condition $|D(w)|^2 = n[\Psi(w)]$. Further, the function

$$g[\Psi(w)] = [\sqrt{\Psi'(w)}\, D(w)]^{-1}$$

has the same differential properties; therefore, if

$$g[\Psi(w)] = \sum_0^\infty \frac{a_k}{w^k}, \quad \text{then } a_k = O\left(\frac{1}{k^{p+\alpha}}\right).$$

Consider the generalized Faber polynomials $B_n(z)$, defined by the expansion

$$\frac{g[\Psi(w)]\Psi'(w)}{\Psi(w) - z} = \sum_0^\infty \frac{B_n(z)}{w^{n+1}}, \quad z \in G,\ |w| > 1.$$

From this we find that, under the conditions considered, for every closed set F

in the domain G there exists a constant $C_1(F)$ such that for all n we have

$$|B_n(z)| \leq \frac{C_1(F)}{n^{p+\alpha}}, \quad z \in F \subset G. \tag{1}$$

Let us consider the behavior of the polynomials $B_n(z)$ outside G. Let $z \in G_\infty$ and lie inside the line Γ_R, which is the image of the circle $|w| = R$. Then, using the definition, we find

$$B_n(z) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{[\Phi(\zeta)]^n g(\zeta)}{\zeta - z}\, d\zeta = g(z)[\Phi(z)]^n + \frac{1}{2\pi i} \int_{\Gamma} \frac{[\Phi(\zeta)]^n g(\zeta)}{\zeta - z}\, d\zeta. \tag{2}$$

Denote the last integral by $\mathcal{E}_n(z)$, and, passing under the integral sign to the variable w, we find that for $\mathcal{E}_n(z)$ an inequality analogous to inequality (1) holds, with the only difference that $F \subset G_\infty$. Therefore, for the polynomials $B_n(z)$ we have the formula

$$B_n(z) = g(z)[\Phi(z)]^n + O\left(\frac{1}{n^{p+\alpha}}\right), \qquad z \in F \subset G_\infty. \tag{3}$$

Lemma. Let Γ be a smooth curve and $\Psi^{(p+4)}(w) \in H_1$; then, for $|w_0| = 1$, the function

$$F(w, w_0) = \frac{\Psi'(w)}{\Psi(w) - \Psi(w_0)} - \frac{1}{w - w_0}$$

is continuously differentiable in $|w| \geq 1$ $(p+1)$ times.

Let $z = \Psi(w) = \sum_{-1}^{\infty} \beta_k w^{-k}$; then, by virtue of the condition $\Psi^{(p+4)}(w) \in H_1$, we have that the series $\sum_1^\infty n^{p+3}|\beta_n|$ converges. Substituting into the preceding formula, we find

$$F(w, w_0) =$$

$$= \frac{\displaystyle\sum_1^\infty \beta_n \frac{(w^{n-1} + w^{n-2}w_0 + \cdots + w_0^{n-1}) + (w^{n-2} + w^{n-3}w_0 + \cdots + w_0^{n-2})w_0 + \cdots + w^{n-1}}{w^{n+1}w_0^n}}{\displaystyle\beta_{-1} - \sum_1^\infty \beta_n \frac{w^{n-1} + w^{n-2}w_0 + \cdots + w_0^{n-1}}{w^n w_0^n}}.$$

The general term of the series standing in the numerator does not exceed the quantity $0.5|\beta_n|n(n+1)$, and in the denominator—the quantity $|\beta_n|n$; moreover, it is not difficult to show that the denominator is different from zero. Thus, for $|w_0| = 1$ the function $F(w, w_0)$ is continuous in the closed domain $|w| \geq 1$. Moreover, from the construction of the series standing in the numerator and denominator it follows that, even after $(p+1)$-fold differentiation, each of them remains uniformly convergent for $|w| \geq 1$. Consequently, the function $F(w, w_0)$ is continuously differentiable $(p+1)$ times, and the lemma is proved.

Theorem 1. Under the conditions considered, imposed on Γ and on $g(z)$, for the generalized Faber polynomials with $z \in \Gamma$ the asymptotic formula holds

$$B_n(z) = g(z)[\Phi(z)]^n \left[1 + O\left(\frac{\ln n}{n^{p+\alpha}}\right)\right]. \tag{4}$$

For the proof it is sufficient in formula (2) to estimate the quantity $\mathcal{E}_n(z)$ for $z \in \Gamma$. We have

$$\mathcal{E}_n[\Psi(w_0)] = \frac{1}{2\pi i} \int_{|w|=1} (w^n g[\Psi(w)])\, F(w, w_0)\, dw - w_0^n \sum_{n+1}^{\infty} \frac{a^k}{w_0^k}. \tag{5}$$

By virtue of the lemma, the first term has order $n^{-p-\alpha}$, while the second, by a theorem of S. M. Nikol' skii, has order $n^{-p-\alpha} \ln n$, and the theorem is proved.

Formulas (1), (3), and (4) completely characterize the behavior of the polynomials $B_n(z)$ in terms of the boundary properties of the function $g(z)$ and the smoothness of the contour Γ.

The polynomials $B_n(z)$ serve for the representation of analytic functions. It is not difficult to show that, under the conditions considered, imposed on the contour Γ and on the function $g(z)$, every function $f(z)$ analytic in the domain G, whose primitive of order p is representable by a Cauchy integral through its angular boundary values, expands in a series in the polynomials $B_n(z)$, converging uniformly inside G and absolutely. The special case $g(z) \equiv 1$ was considered in paper ([2]); the general case is proved analogously.

Analogous results also hold for the orthogonal polynomials $P_n(z)$, if the contour Γ and the weight $n(z)$ satisfy the conditions mentioned.

Theorem 2. For every closed set F in the domain G there exists a constant $C_2(F)$ such that, for all n and $z \in F$, the inequality

$$|P_n(z)| \leq \frac{C_2(F)}{n^{p+\alpha}}. \tag{6}$$

holds.

The proof of Theorem 2 in the special case when Γ is a regular analytic curve is given in paper ([3]). Taking into account the estimate for the first term of the right-hand side of (5), we easily obtain the proof in the case of a smooth boundary.

Theorem 3. For the orthogonal polynomials there holds, uniformly inside G_∞, the asymptotic formula

$$P_n(z) = g(z)[\Phi(z)]^n \left[1 + O\left(\frac{1}{n^{p-\alpha}}\right)\right], \qquad z \in F \subset G_\infty. \tag{7}$$

Let μ_n be the leading coefficient of the polynomial $P_n(z)$ and $\Phi(z) = \gamma z + \gamma_0 + \cdots$; then, with the aid of (2), we find

$$g(z)[\Phi(z)]^n + \mathcal{E}_n(z) = \frac{a_0\gamma^n}{\mu_n} P_n(z) + a_0\gamma^n Q_{n-1}(z). \tag{8}$$

Under the condition that Γ is a regular analytic curve, the following relations were established in paper ([3]):

$$\frac{a_0^2\gamma^{2n}}{\mu_n^2} = 1 + O\left(\frac{1}{n^{2p+2\alpha}}\right), \tag{9}$$

$$\frac{a_0^2\gamma^{2n}}{2\pi} \int_\Gamma n(z)|Q_{n-1}(z)|^2\,|dz| = O\left(\frac{1}{n^{2p+2\alpha}}\right). \tag{10}$$

Using the lemma, it is not difficult to show that (9) and (10) are also valid in the case under consideration. Further, with the aid of (10), for the polynomial $a_0\gamma^n Q_{n-1}(z)$ divided by $[\Phi(z)]^n$, an inequality analogous to (6) is established for $z \in F \subset G_\infty$, and Theorem 3 is proved.

Theorem 4. For the orthogonal polynomials, for $z \in \Gamma$, the asymptotic formula

$$P_n(z) = g(z)[\Phi(z)]^n \left[1 + O\left(\frac{\sqrt{n}}{n^{p-\alpha}}\right)\right] \tag{11}$$

holds.

Since for $z \in \Gamma$ we have $\mathcal{E}_n(z) = O\left(\frac{\ln n}{n^{p+\alpha}}\right)$, by virtue of (8) it is enough to estimate the polynomials $Q_{n-1}(z)$ on Γ. With the aid of inequality (10), analo-

similarly to how this is done in paper (1), where the case $p = 0$ is considered, we find that

$$a_0^r Q_{n-1}(z) = O\left(\frac{\sqrt{n}}{n^{p+\alpha}}\right),$$

and therefore the theorem is proved.

Formulas (6), (7), (11) characterize the dependence of the orthogonal polynomials on the differential properties of the weight and on the smoothness of the contour. In all the formulations it was assumed that $\alpha < 1$; if, however, $\alpha = 1$, then in the assertions, instead of α, one should put any $\alpha' < 1$, with a corresponding choice of the constant.

Theorem 5. *Under the conditions considered, imposed on the contour* Γ *and the weight* $n(z)$, *every function* $f(z)$ *analytic in the domain* G *whose primitive of order* p *is representable by a Cauchy integral through its boundary values expands in a series in the orthogonal polynomials, converging uniformly inside* G.

The proof of Theorem 5 is analogous to the proof of Theorem 1 from paper (4), taking into account the above-mentioned estimates for the quantity $\mathcal{E}_n(z)$.

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Note: Figure translations are in progress. See original paper for figures.