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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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MATHEMATICS

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GEOMETRIC PROPERTIES OF SOLUTIONS OF NONLINEAR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

In solving a number of important qualitative problems of continuum mechanics and analysis, one makes essential use of the geometric properties of solutions of systems of first-order partial differential equations

$$F_i(x,\ y,\ u,\ v,\ \partial u/\partial x,\ \partial u/\partial y,\ \partial v/\partial x,\ \partial v/\partial y) = 0 \quad (i = 1,\ 2). \tag{1}$$

In the case of *linear* systems of elliptic type, the geometric properties of solutions have been studied in detail. This was done in papers ([1−5]), etc.; moreover, in recent years the strongest results and methods in the indicated direction have been obtained by I. N. Vekua.

In the present paper we shall note a number of new geometric properties of solutions of the so-called *strongly elliptic nonlinear systems*, whose definition was given in papers ([6,7]). Let us recall this definition, slightly generalizing it.

Consider an arbitrary mapping, differentiable at the point z_0,

$$w = f(z) = u(x,\ y) + iv(x,\ y), \tag{2}$$

whose Jacobian is different from zero at this point. The principal linear part of this mapping at the point z_0 transforms into a square with vertex $w_0 = f(z_0)$, of side 1, inclined at an angle β to the u-axis, the parallelogram with vertex z_0, whose characteristics $V_\beta,\ W_\beta,\ \alpha_\beta$, and θ_β are indicated in Fig. 1. Since the partial derivatives $\partial u/\partial x, \dots, \partial v/\partial y$ are expressed elementarily in terms of these characteristics, system (1) may formally be rewritten in the form

Fig. 1

$$G_{\beta}^{(i)}(x,\ y,\ u,\ v,\ V_{\beta},\ W_{\beta},\ \alpha_{\beta},\ \theta_{\beta}) = 0 \quad (i = 1,\ 2). \tag{3}$$

We shall call system (1) or (3) *strongly elliptic* in a domain D if, for any of its solutions (2), at any point (x, y) at which the Jacobian $J = \partial(u, v)/\partial(x, y) \neq 0$, this system can be rewritten in the form:

$$W_{\beta} = W_{\beta}(x,\ y,\ u,\ v,\ V_{\beta},\ \alpha_{\beta}), \qquad \theta_{\beta} = \theta_{\beta}(x,\ y,\ u,\ v,\ V_{\beta},\ \alpha_{\beta}), \tag{4}$$

where the functions W_{β} and θ_{β} are single-valued and differentiable, and there exists a constant $\delta > 0$ such that for all values of the arguments we have

$$\delta < \theta_{\beta} < \pi - \delta, \qquad \frac{\partial W_{\beta}}{\partial V_{\beta}} > \delta. \tag{5}$$

A solution (2) of the strongly elliptic system (1) will be called a *quasiconformal* mapping corresponding to this system. Using the methods of the theory of quasiconformal mappings developed in papers ([6,7,2]), as well as the topological methods of papers ([8–10]), one can prove the following theorems.

Theorem 1. For every Riemann surface F of hyperbolic type and for every strongly elliptic system (1), there exists a homeomorphic quasiconformal mapping $w = f(z)$ of the surface F onto the unit disk corresponding to this system.

Theorem 2. Every differentiable solution $w = f(z)$ in a domain D of the strongly elliptic system (1), for which $u \neq$ const, $v \neq$ const, realizes a homeomorphic mapping of D onto some Riemann surface.

From these theorems the following theorem follows:

Theorem 3. Every differentiable solution $w = f(z)$ in a domain D of the strongly elliptic system (1), for which $u \neq$ const, $v \neq$ const, can be represented as the superposition of an analytic function $w = F(\omega)$ and a homeomorphic quasiconformal mapping $\omega = T(z)$ of the domain D onto itself.

The last theorem expresses the fact that in the domain D one can introduce curvilinear coordinates in which the given solution $f(z)$ of the strongly elliptic system will be an analytic function.

These theorems make it possible to extend to solutions of strongly elliptic systems all the topological properties of analytic functions (the maximum principle, the boundary correspondence principle, the argument principle, Rouché' s theorem, etc.).

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Note: Figure translations are in progress. See original paper for figures.