BENDING OF A PRISMATIC BAR WEAKENED BY A CIRCULAR CAVITY

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

THEORY OF ELASTICITY

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BENDING OF A PRISMATIC BAR WEAKENED BY A CIRCULAR CAVITY

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1. Consider the bending of a homogeneous isotropic prismatic bar with a circular cavity under the action of a concentrated force applied along one of the minor axes of the square (Fig. 1).

As is known ([1]), determination of the stress state in the bending of a bar reduces to finding a function $\varphi(z)$ of the complex variable z, regular in the domain S, from the boundary conditions:

$$\varphi_j(t) + \overline{\varphi_j(t)} = 2F_j(t) + D_j \qquad (j = 1, 2),$$

where t is the affix of a point of the square L_1 or of the circle L_2; D_1 and D_2 are real constants; one of them, for example D_2, is set equal to zero;

$$F_j(t) = -\left(1 - \frac{\sigma}{2}\right)\frac{y^3}{3} - \frac{\sigma}{2}x^2 y + 2(1+\sigma)\int xy\,dx \qquad (j = 1, 2); \qquad (1)$$

σ is Poisson' s ratio.

Fig. 1

2. Take the function mapping the exterior of the curvilinear square onto the exterior of the unit circle γ in the form

$$Z = A\left(\zeta - \frac{m}{\zeta^3}\right)\left(A = \frac{a+b}{2},\ m = \frac{a-b}{a+b} = \frac{1}{9}\right). \qquad (2)$$

From equalities (1), after some simple transformations, we obtain:

$$F_1(t) = F^*(\tau) = A^3 i\{C_1(\tau - \tau^{-1}) + C_3(\tau^3 - \tau^{-3}) + C_5(\tau^5 - \tau^{-5}) +$$

$$+ C_7(\tau^7 - \tau^{-7}) + C_9(\tau^9 - \tau^{-9})\} \quad \text{on } \gamma; \tag{3}$$

$$F_2(t) = iR^3 \left\{ e_1 \left(\frac{t}{R} - \frac{R}{t} \right) + e_3 \left(\left(\frac{t}{R} \right)^3 - \left(\frac{R}{t} \right)^3 \right) \right\} \quad \text{on } L_2. \tag{4}$$

Here τ is the affix of a point of γ; t is the affix of a point of L_2;

$$e_1 = \frac{3}{8} + \frac{1}{4}\sigma; \qquad e_3 = -\frac{1}{8}$$

$$C_1 = \frac{1}{8}\left[3(1-m) - 10m^2 + 2\sigma(1-6m^2)\right];$$

$$C_3 = -\frac{1}{24}\left[3 - 2m - 9m^3 + 2m\sigma(2-3m^2)\right];$$

$$C_5 = \frac{m}{8}\left[3m + \frac{1}{5} + \frac{6}{5}\sigma\right], \qquad C_7 = -\frac{m^2}{56}(5-2\sigma),$$

$$C_9 = -\frac{1}{8}m^3.$$

3. The required function $\varphi(z)$, regular in the domain occupied by the cross-section of the bar, will be sought in the form

$$\varphi(z) = \varphi_1(z) + \varphi_2(z), \tag{5}$$

where $\varphi_1(z)$ is a function regular inside L_1; $\varphi_2(z)$ is a function regular outside L_2, and, moreover, $\varphi_2(\infty) = 0$.

The function $\varphi_2(z)$ can be represented in the form of the series

$$\varphi_2(z) = \sum_{k=1}^{\infty} b_k^{(2)} \left(\frac{R}{Z} \right)^k; \tag{6}$$

here the unknown coefficients $b_k^{(2)}$ shall be taken as purely imaginary.

D. I. Sherman indicated a method for expanding the function $\varphi_1(z)$ in a series in polynomials ([2]). We reproduce these arguments here as applied to the mapping function (2) that has been adopted.

Expand the boundary value of the function $\varphi_1(z)$ on L_1 in a Fourier series on the mapped circle γ; putting $\varphi_1(t) = \varphi_1^*(\tau)$, we shall have

$$\varphi_1^*(\tau) = \sum_{k=-\infty}^{\infty} b_k^{(1)} \tau^k. \tag{7}$$

Here the unknown coefficients $b_k^{(1)}$ $(k \neq 0)$ are taken to be purely imaginary, while the constant $b_0^{(1)}$ is real.

Expanding the function $g(z)$, inverse to the function (2), in a Laurent series and raising it to the power k, we obtain:

$$g^k(z) = \left(\frac{z}{A}\right)^k \sum_{n=0}^{\infty} a_n^{(k)} \left(\frac{A}{z}\right)^{4n}. \tag{8}$$

Here

$$a_0^{(k)} = a_0^k = 1, \qquad a_n^{(k)} = \frac{1}{na_0} \sum_{k_1=1}^{n} \{k_1(k+1) - n\} a_{k_1} a_{n-k_1}^{(k)}, \quad n \geq 1$$

$(a_1 = m,\ a_2 = -3m^2,\ a_3 = 15m^3,\ a_4 = -91m^4,\ a_5 = 612m^5,\ a_6 = -4389m$, etc.).

Substituting (8) into (7), on the basis of Cauchy' s integral we shall have:

$$\varphi_1(z) = b_0^{(1)} + \sum_{\nu=0}^{\infty} H_\nu \left(\frac{z}{R}\right)^\nu, \tag{9}$$

where it has been put

$$H_\nu = \left(\frac{R}{A}\right)^\nu \sum_{k=E(\nu)}^{\infty *} a_{\frac{k-\nu}{4}}^{(k)} b_k^{(1)} \tag{10}$$

$(E(\nu) = \nu\ (\nu \neq 0);\ E(\nu) = 4\ (\nu = 0))$. The asterisk over the summation sign in (10) indicates that the index k changes by 4 in passing to the adjacent value.

On the basis of the boundary condition on L_2,

$$b_0^{(1)} = 0; \qquad H_\nu - b_\nu^{(2)} = 2iR^3 e_\nu. \tag{11}$$

Here $e_\nu = 0$ for $\nu \neq 1, 3$.

With the aid of (2) and (6), we obtain on L_1

$$\varphi_2(t) = \varphi_2^*(\tau) = \sum_{\varepsilon=1}^{\infty} T_\varepsilon \tau^{-\varepsilon}, \tag{12}$$

where

$$T_\varepsilon = \sum_{k=\varepsilon-4E(\frac{\varepsilon}{4})}^{\infty *} (-1)^{\frac{\varepsilon-k}{4}} \left(\frac{R}{A}\right)^k m^{\frac{\varepsilon-k}{4}} C_{-k}^{\frac{\varepsilon-k}{4}} b_k^{(2)}. \tag{13}$$

Proceeding from the boundary condition on L_1 and formulas (7), (12):

$$-T_\nu + b_\nu^{(1)} - b_{-\nu}^{(1)} = 2iA^3 C_\nu \quad (C_\nu = 0,\ \nu \neq 1,3,5,7,9);$$

$$D_1 = 0. \tag{14}$$

On the basis of (9) one can obtain the expression for the coefficients $b_{-\nu}^{(1)}$ in terms of the coefficients $b_\nu^{(1)}$:

$$b_{-\nu} = \sum_{k=\beta+\frac{\nu+\beta}{3}}^{\infty} {}^*\lambda_{k,\nu} b_k^{(1)} \qquad (\nu = 1, 2, ...), \tag{15}$$

where

$$\lambda_{k,\nu} = \sum_{k_1=\beta+\frac{\nu+\beta}{3}}^{k} {}^*(-1)^{\frac{k_1+\nu}{4}} C_{k_1}^{\frac{k_1+\nu}{4}} m^{\frac{k_1+\nu}{4}} d_{\frac{k-k_1}{4}}^{(k)} \qquad (\beta = 0,1,2).$$

The derivation of this expression was communicated to us by D. I. Sherman as an addition to the expansion, contained in paper ([2]), of a function regular inside L_1 in a series of polynomials.

Substituting $b_\nu^{(2)}$ from (11) into (13) and changing the order of summation with account of (14) and (15), we obtain

$$\sum_{e=1}^{\infty} q_{e,\nu} b_e^{(1)} = f_\nu. \tag{16}$$

Here

$$q_{e,\nu} = d_{e,\nu}, \qquad e = \nu - 4E\left(\frac{\nu}{4}\right) + 4\varepsilon;$$

$$q_{e,\nu} = -\lambda_{e,\nu}, \qquad e = \beta + \frac{\nu+\beta}{3} + 4\varepsilon$$

$$\left(\varepsilon = 0, 1, \ldots, \infty; \quad E\left(\frac{\nu}{4}\right) \text{ is the greatest integer contained in } \frac{\nu}{4}\right);$$

$$d_{e,e} = 1 + d^*_{e,e}; \qquad d_{e,\nu} = d^*_{e,\nu} \quad (e \neq \nu);$$

$$d^*_{e,\nu} = -\sum_{k=e-4E(\frac{e}{4})}^{N(e,\nu)} {}^*(-1)^{\frac{\nu-k}{4}} m^{\frac{\nu-k}{4}} C_{-k}^{\frac{\nu-k}{4}} \left(\frac{R}{A}\right)^{2k} a^{(e)}_{\frac{2e-k}{4}};$$

$$N(e,\nu) = \begin{cases} e, & e \leq \nu, \\ \nu, & e > \nu. \end{cases}$$

$$f_\nu = -2iA_3 \left\{ (-1)^{\frac{\nu-j}{4}} \left(\frac{R}{A}\right)^{j+3} m^{\frac{\nu-j}{4}} C_{-j}^{\frac{\nu-j}{4}} e_j - C_\nu \right\} \qquad (\nu = 1, 3, 5, \ldots). \quad (17)$$

It follows from (17) that for even ν the system (16) represents a homogeneous system of equations separately with respect to $b_\nu^{(1)}$ ($\nu = 0, 2, 4, \ldots$). By virtue of the uniqueness of the solution, they should be set equal to zero. From the system (16) the first 10 equations were separated. In the first equation the sum of the moduli of the coefficients, for the very unfavorable case owing to the nearness of the boundaries ($R/b = 0.9$), is greater than unity; in the remaining equations this sum is less than unity.

Having separated the first equation, the remaining equations are solved by the method of successive approximations; in this, 5 approximations were needed in order to arrive at a very accurate solution.

Taking into account (6), (9), and the fact that $b_0^{(1)} = 0$, we obtain

$$\varphi(z) = \sum_{k=1}^{\infty}{}^* \left\{ H_k \left(\frac{z}{R}\right)^k + b_k^{(2)} \left(\frac{R}{z}\right)^k \right\}.$$

By direct substitution it is easy to verify that the function $\varphi(z)$ found exactly satisfies the boundary condition on L_2. The values $\Delta\%$,

characterizing the degree of accuracy with which the function $\varphi(z)$, found by means of the solution of the truncated system of equations, satisfies the boundary condition at the most characteristic points $t = ib$ and $t = ae^{i\pi/4}$, are respectively equal to:

$$\Delta_1\% = \frac{\varphi(ib) - \overline{\varphi(ib)} - 2F_1(ib)}{2F_1(ib)}\,100\% = -0.05146\%,$$

$$\Delta_2\% = 0.02222\%.$$

Thus, the boundary condition is satisfied with sufficient accuracy also on L_2. The magnitudes of the shear stresses acting at the points ib, $i\frac{R+b}{2}$, and iR of the neutral axis of the section, for $\sigma = 0.3$ and $R/A = 0.8$, are:

$$X_{z,1} = 2.1565\,\frac{Pb^2}{I}, \qquad X_{z,2} = 2.1761\,\frac{Pb^2}{I}, \qquad X_{z,3} = 2.2563\,\frac{Pb^2}{I}.$$

Here I is the moment of inertia of the section with respect to the neutral axis.

The magnitudes of the stresses obtained by D. I. Zhuravsky' s formula ($\tau = 2.1888Pb^2/I$) at the point iR are 1.49% smaller than those obtained by our formulas, while at the points $i\frac{b+R}{2}$ and ib they are larger, respectively, by 0.58 and 2.99%; i.e., the stress determined by Zhuravsky' s formula is sufficiently close to the true value.

The magnitudes of the shear stresses acting at the indicated points, obtained by us on the electric model (EM-7), are:

$$X_{z,1} = 2.11\,\frac{Pb^2}{I}, \qquad X_{z,2} = 2.17\,\frac{Pb^2}{I}, \qquad X_{z,3} = 2.35\,\frac{Pb^2}{I}.$$

The discrepancies between the shear stresses obtained by our formulas and on the EM-7 are: at the point ib, 2.17%; at the point $i\frac{b+R}{2}$, 0.03%; at the point iR, 4.2%.

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REFERENCES

1. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, 1954. D. I. Sherman, *Izv. Acad. Sci. USSR, OTN*, No. 7 (1951).

Note: Figure translations are in progress. See original paper for figures.