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Abstract

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ON THE NUMBER OF LIMIT CYCLES OF THE EQUATION $\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$, WHERE P AND Q ARE POLYNOMIALS OF DEGREE n

The note is devoted to the problem of giving an upper estimate for the number of limit cycles of the equation

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}, \tag{1}$$

where P and Q are polynomials of degree n. It is shown that the number of limit cycles of such an equation does not exceed $\frac{9n^3+n^2-6n+6}{2}$ for $n = 2k-1$, and $\frac{9n^3-4n^2-27n+14}{2}$ for $n = 2k$.

The method by which this estimate is obtained is similar to the method used by us in estimating the number of limit cycles of equations of the form (1), when the degrees of the polynomials P and Q were equal to two ([1]).

Consider the complex equation

$$\frac{dy}{dx} = \frac{\widetilde{P}(x,y)}{\widetilde{Q}(x,y)}; \tag{2}$$

$\widetilde{P}$ and $\widetilde{Q}$ are polynomials of degree n with, generally speaking, complex coefficients. The space (x, y) is the complex projective space. We shall denote the corresponding real four-dimensional space by R_4. In R_4 we introduce, in some manner, a compact metric.

By a solution $y = \varphi(x)$ of equation (2) we shall understand a complete analytic function (the Weierstrass analytic continuation of a solution element). An integral curve will mean the graph Φ of the solution $y = \varphi(x)$ in R_4. In addition to ordinary points, we shall also assign to Φ poles and branch points of finite order.

A closed one-dimensional curve (a topological circle lying on an integral curve Φ) will be called a cycle.

Now consider the real equation

$$\frac{dy^*}{dx^*} = \frac{P(x^*, y^*)}{Q(x^*, y^*)}; \tag{2*}$$

x^* and y^* are real variables; P and Q are polynomials of degree n with real coefficients. The integral curves of this equation may be considered as the result of intersecting the real plane ($\operatorname{Im} x = 0$, $\operatorname{Im} y = 0$) with the integral curves of the complex equation (2) in the case where the coefficients of the polynomials appearing on the right-hand side take the given real values and $\operatorname{Re} x = x^*$, $\operatorname{Re} y = y^*$. We shall say that such a complex equation (2) corresponds to the given real equation (2^*).

Lemma 1. *A limit cycle of the real equation* (2^*) *is a cycle not homologous to zero on the integral curve of the corresponding complex equation (2).*

Two limit cycles of the real equation (2^*) *lying on one and the same integral curve of the corresponding complex equation (2) are not homologous to each other.*

We shall call a cycle L lying on an integral curve of equation (2) **simple** if its projections onto the x- and y-planes have no self-intersections. We shall say that cycles $L_1, \ldots, L_m$, lying on integral curves of equation (2), are **regularly arranged** if their projections onto the x- and y-planes do not intersect pairwise.

Lemma 2. *The maximal number of limit cycles of equation (1) is not greater than the maximal number of simple, regularly arranged cycles of equation (2) that are not homologous to zero and not homologous to one another.*

Denote by D the space of the coefficients of the polynomials P and Q of equation (2) [an $(n+1)(n+2)$-dimensional complex space, or a $2(n+1)(n+2)$-dimensional real space].

Lemma 3. *There exists a set $N \subset D$, not separating the space D, such that for any point $\alpha \in D \setminus N$ the number of simple, regularly arranged cycles of equation (1) that are not homologous to zero and not homologous to one another is constant.*

Lemmas 1, 2, 3 are contained in the paper ([1]).

Lemma 4. *Let $n = 2k - 1$. There exists a domain U in the space D, for every point of which there are no more than*

$$\frac{9n^3 + n^2 - 6n + 6}{2}$$

regularly arranged simple cycles that are not homologous to zero and not homologous to one another.

The proof of this lemma is carried out as follows. Consider the equation

$$\frac{dy}{dx} = \frac{P_0(x,y)}{Q_0(x,y)}, \tag{3}$$

whose general solution is the pencil of polynomials

$$M(x,y) = cN(x,y) \tag{4}$$

of degree k, i.e. (up to a common factor)

$$P_0(x,y) = M'_{xN} - N'_{xM},$$

$$Q_0(x,y) = M'_{yN} - N'_{yM}.$$

Let equation (3) be such that all its singular points are distinct. Let this equation correspond to a point $\alpha^* \in D$. Let A be some $(n+1)(n+2)$-dimensional complex vector. Consider the complex line $\alpha = \alpha^* + A\tau$ in the space D.

Let L be a closed curve on a solution of equation (3). Let $(x_0, y_0) \in L$, and let $y(x, y^*, \tau)$ be the solution of equation (2), corresponding to the point $\alpha^* + A\tau$, passing through the point (x_0, y^*). Let C be the projection of L onto the x-plane.

Consider the increment of the function

$$\left.\frac{\partial y(x, y^*, \tau)}{\partial \tau}\right|_{\substack{y^*=y_0 \\ \tau=0}}$$

when going around L. This increment, up to a factor, is equal to

$$\int_C \exp\left[-\int_{x_0}^{x} \left(\frac{P_0}{Q_0}\right)'_y dx\right] \left(\frac{P_0}{Q_0}\right)'_\tau dx. \tag{5}$$

Let c be the value of the parameter in (4) for which the curve of the pencil passes through the point (x_0, y_0). For fixed curve C on the x-plane, (5) is a function of c. Denote it by $v_C^{(c)}$.

Similarly to how it is done in [^1], one can show that in order to estimate the number of simple, correctly situated cycles, not homologous to zero and not homologous to one another, in a neighborhood of the equation α^*, it is sufficient to estimate the maximum possible number s of closed curves $L_1, \dots, L_s$ lying on the complex curves (4), with the singular points of equation (3) removed from them, such that:

a) L_i $(i = 1, \dots, s)$ are not homologous to zero on the corresponding curve from (4) with the singular points of equation (3) removed;

b) L_i and L_j, lying on one and the same curve, are not homologous to one another;

c) the projections of L_i $(i = 1, \dots, s)$ onto one of the coordinate planes x or y have no self-intersections. Let $L_1, \dots, L_r$ be those curves whose projections onto the x-plane have no self-intersections, and $L_{r+1}, \dots, L_s$ those curves whose projections onto the y-plane have no self-intersections. Then the projections of $L_1, \dots, L_r$ onto the x-plane do not intersect one another, and the projections of $L_{r+1}, \dots, L_s$ onto the y-plane do not intersect one another;

d) for the curves $L_1, \dots, L_r$ one has

$$\gamma_C^{(c)} = 0; \tag{6}$$

for the curves $L_{r+1}, \dots, L_s$ the symmetric equality holds (obtained by replacing x by y and P_0 by Q_0).

After the corresponding computations, we find from (5)

$$\gamma_C^{(c)} = \int_C \frac{P_1(x,y)Q_0(x,y) - Q_1(x,y)P_0(x,y)}{Q_0(x,y)M^2(x,y)}\, dx; \tag{7}$$

P_1 and Q_1 are polynomials of degree $2k-1$, whose coefficients are the components of the vector A. Thus the question reduces to finding the number of roots of equation (7) (with respect to c) for various C (or of the analogous equation obtained by replacing x by y, P_0 by Q_0, and M by N).

In order to do this, we shall use the following additional consideration. Suppose that the pencil of polynomials (4) depends analytically on the complex parameter η and, for $\eta = 0$, turns into the pencil

$$M_0(x,y) = cN_0(x,y),$$

for which equation (3) may have multiple singular points. Let $C(\eta)$ be a curve in the x-plane (or y-plane), varying continuously with η and, for $\eta \neq 0$, not passing through the projections of the singular points of equation (3) onto the x-plane (or y-plane). Let $\gamma_{C(\eta)}(c,\eta)$ be expression (7), written for the pencil η and the curve $C(\eta)$. Suppose, furthermore, that $\gamma_{C(\eta)}(c,\eta)$ is a function analytic in a neighborhood of $\eta = 0$. Then the number of zeros of $\gamma_{C(\eta)}(c,\eta)$ as a function of c, for sufficiently small η, obviously coincides with the number of zeros of $\gamma_{C(0)}(c,0)$. Thus, when considering equation (3) with multiple singular points, one may regard as zeros of the function $\gamma_{C(0)}(c,0)$ those curves in the x-plane (or y-plane) which may pass through singular points and whose projections intersect, but which are limits as $\eta \to 0$ of curves not passing through singular points and with nonintersecting projections.

Consider the following specific pencil:

$$y^k - \varepsilon = c(x^k - 1).$$

It corresponds to the equation

$$\frac{dy}{dx} = \frac{x^{k-1}(y^k - \varepsilon)}{y^{k-1}(x^k - \varepsilon)}.$$

This equation has the following singular points: k^2 points $(\sqrt[k]{1}, \sqrt[k]{\varepsilon})$; a $(k-1)^2$-fold point $(0,0)$, through which the solution $c = \varepsilon$ passes; a k^2-fold infinitely distant point, through which the solution $c = 0$ passes, and a k^2-fold infinitely distant point, through which the solution $c = \infty$ passes.

The curves L, lying on the solutions $c \neq \varepsilon, 0, \infty$, are found by computing the number of zeros of the integral (7) for the curves C of the x-plane encircling various combinations of the points $\sqrt[k]{1}$ and $\sqrt[k]{1 - \varepsilon/c}$. For this purpose (7) is found separately for curves encircling only the points $\sqrt[k]{1}$, by means of residues, after which one passes to the limit as $\varepsilon \to 0$ and (7) is computed for curves encircling pairs of coalesced points $\sqrt[k]{1}$ and $\sqrt[k]{1 - \varepsilon/c}$.

In all, the number of curves with the required properties and with nonintersecting projections on the solutions $c \neq \varepsilon, 0, \infty$ turns out to be $(9k-5)(2k-1)^2$. In addition, on the solution $c = \varepsilon$ there are $(2k-3)^2$ of them, and on the solutions $c = 0$ and $c = \infty$ there are another $(2k-1)^2$ on each.

Thus, on all solutions there are

$$(9k-5)(2k-1) + (2k-3)^2 + 2(2k-1)^2$$

curves. Through each singular point of equation (2) (with the exception of a set of equations whose dimension is two real units less than the dimension D) there pass two singular solutions for which this point is ordinary. On such a solution, in a neighborhood of the singular point, there lies a cycle homologous to zero but encircling this point. Equation (2) has $(2k-1)^2$ finite singular points and $2k$ infinite ones. One of the singular solutions passing through each infinite singular point is the infinitely distant straight line. Consequently, from the number found one must exclude $2(2k-1)^2 + 2k$ curves corresponding to cycles homologous to zero, and we obtain the required number.

Lemma 5. Let $n = 2k$. There exists a domain U in the space D such that for each point of it there are no more than

$$\frac{9n^3 - 4n^2 - 27n + 14}{2}$$

simple, properly arranged cycles, not homologous to zero and not homologous to one another.

The proof of this lemma is analogous to the proof of Lemma 4.

From Lemmas 2, 3, 4, and 5 we obtain the theorem:

Theorem. For an equation of the form (1) there exist no more than

$$\frac{9n^3 + n^2 - 6n + 6}{2}$$

limit cycles if n is odd, and no more than

$$\frac{9n^3 - 4n^2 - 27n + 14}{2}$$

limit cycles if n is even.

We note that, as was shown by N. F. Otrokov[2], there exist equations of the form (1) with the number of limit cycles

$$\frac{n^2 + 5n - 14}{2}$$

and

$$\frac{n^2 - 5n - 26}{2},$$

respectively for even and odd n.

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[1] I. G. Petrovskii, E. M. Landis, *Matem. sbornik*, **37** (79), No. 2, 209 (1955).
[2] N. F. Otrokov, *Matem. sbornik*, **34** (76), No. 1 (1954).

Note: Figure translations are in progress. See original paper for figures.