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# MATHEMATICS

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**Abstract**

**Full Text**

## MATHEMATICS

**A. KHAIMOVICH**

### ON SOME APPLICATIONS OF A THEOREM OF F. RIESZ

*(Presented by Academician S. L. Sobolev, 26 X 1956)*

F. Riesz' s theorem has been applied to many problems of functional analysis. Here we shall give some applications concerning functional equations.

§ 1. Let  $A$  be a linear operator whose domain of definition  $D_A$  is a subspace of some Hilbert space  $L_p(\Omega)$ , where  $\Omega$  is the domain of definition of the functions forming this subspace. Let  $D$  be the range of the operator  $A$ ; consider the equation

$$Au(P) = f(Q), \quad u(P) \in D_A, \quad f(Q) \in D, \quad P \in \Omega. \quad (1)$$

Assume that, when  $f(Q)$  is given, equation (1) has a unique solution in  $D_A$ . Then one may assert that the value  $u(P)$  at a certain point  $P_0$  of  $\Omega$  is a functional  $\Phi$ , additive and homogeneous with respect to  $f(Q)$ . If, moreover,

$$|u(P_0)| \leq M \|f(Q)\|_{L_p}, \quad (2)$$

then the functional  $\Phi$  is linear. It then follows from F. Riesz' s theorem that there exists a function of  $P$ , say  $R_{P_0}(P) = R(P, P_0)$ , belonging to  $L_p$  and such that

$$u(P_0) = \int_{\Omega} R(P_0, P) f(P) dP, \quad (3)$$

and the general solution of equation (1) is written in this form.

**Special cases. A.** If the operator  $A$  is the Laplacian  $\Delta$ ,  $f$  is a continuous function, and  $u(P)$  belongs to the class of functions defined on  $\Omega$ , twice differentiable and vanishing on the boundary of  $\Omega$  (the Dirichlet problem), then equation (1) is Poisson' s equation

$$\Delta u = f(Q);$$

then inequality (2) can be proved. In this case  $R$  is the Green's function, whose existence is thus proved directly.

**B.** If the operator  $A$  is the integral operator

$$u(P) + \lambda \int_{\Omega} K(P, Q)u(Q) dQ,$$

where  $K$  is a Carleman kernel and  $u(P)$  is a function belonging to  $L_2$ , then it is easy to prove that

$$|u(P_0) - f(P_0)| \leq M\|u(P)\|_{L_2}.$$

In the case of a Hilbert-Schmidt kernel <sup>(5)</sup>, this inequality is valid for every point  $P_0$  of  $\Omega$ , except possibly for a set of measure zero.

Using one more known theorem on completely continuous operators ([1], p. 177), we obtain

$$|u(P) - f(P)| \leq M_1\|f(P)\|_{L_2}$$

and, consequently, (3). In this case the function  $R$  is the resolvent of  $\Gamma(P_0, P; \lambda)$ .

In both cases the characteristic properties of the Green function and of the resolvent are obtained directly.

§ 2. Let now  $A$  be a linear operator whose domain of definition  $D_A$  is a subspace of some Banach function space. As in § 1,  $u(P_0)$ , satisfying (1), is an additive and homogeneous functional of  $f(P)$ . If, in addition,

$$|u(P_0)| \leq M\|f(Q)\|_B,$$

then, again by Riesz' s theorem, there exists a measure  $R_{P_0}(P)$  such that

$$u(P) = \int_{\Omega} f(P) dR_{P_0}. \quad (3')$$

**Special cases.** A. Let  $A$  be a second-order differential operator

$$\sum_{h,k=1}^n a_{hk}(P) \frac{\partial^2 u}{\partial x_h \partial x_k} + \sum_{i=1}^n b_i(P) \frac{\partial u}{\partial x_i} + c(P)u \equiv Au, \quad (4)$$

which satisfies the conditions: 1)  $\Omega$  is bounded; 2) one can define a function  $\omega(P)$ , regular with respect to  $A$ , and such that  $A\omega < 0$ ; 3) there exists a function  $w(P)$ , nonnegative, regular in  $\Omega + \text{fr } \Omega$ , for which  $Aw \leq -1$ . Then for

every solution of (4), regular in  $\Omega + \text{fr}\Omega$ , equal to zero on the boundary of the domain  $\Omega$ , we have

$$|u(P)| \leq w(P) \sup_{\Omega} |f| = w(P) \|f\|_B$$

(see [3], p. 694). In this case the existence of the measure  $R$  holds.

B. If  $A$  is the linear operator defined above, and the class  $D_A$  consists of functions  $u$  satisfying the condition  $\partial u / \partial \nu + bu = 0$  and continuous on the boundary  $\Omega$  (see [2]), where the coefficient  $c$  is negative, then we have

$$|u(P)| \leq \frac{1}{\min |c|} \|f\|_B.$$

Then, as above, the measure  $R$  exists.

One can prove that the measure  $R$ , depending on the parameter  $P_0$ , has the following properties: it is equal to zero when  $P_0$  belongs to the boundary  $\Omega$ , and it can be written in the form

$$dR = \varphi(P_0, P) dP,$$

where  $\varphi$  is a function satisfying the condition  $A\varphi = 0$  for  $P \neq P_0$ , while for  $P = P_0$  it has a singularity of the same type as the elementary solution (in the sense of L. Schwartz (see [4], p. 126 and following)).

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*Note: Figure translations are in progress. See original paper for figures.*

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