E. N. ROZENWASSER

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Abstract

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E. N. ROZENWASSER

ON THE STABILITY OF NONLINEAR CONTROL SYSTEMS

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In the present note, on the basis of a theorem of A. I. Lur' e (^{1}), sufficient conditions of stability "in the large" are obtained for certain nonlinear control systems described by differential equations of the fifth and sixth orders. The criteria given make it possible effectively to solve the problem of the stability "in the large" of the established motion of a number of practically important automatic control systems.

Below, for formulas and notation from (^{1}), the numbering adopted there is retained.

Let the control system under consideration be described by the equations ((^{1}), Ch. I, (1.19))

$$\dot{\eta}_k = \sum_{\alpha=1}^{n} b_{k\alpha}\eta_\alpha + h_k f(\sigma) \quad (k = 1, 2, \dots, n);$$

$$\sigma = \sum_{s=1}^{n} j_s \eta_s \quad (n = 5, 6). \tag{1}$$

We assume that the characteristic equation of the open-loop system

$$D(\lambda) = |b_{k\alpha} - \lambda\delta_{k\alpha}| = 0 \tag{2}$$

($\delta_{k\alpha}$ is the Kronecker symbol) has roots $\lambda = \lambda_\rho$ ($\rho = 1, 2, \dots, n;\ n = 5, 6$), distinct from zero and having negative real parts.*

The characteristic equation of the linearized system (1), where $f(\sigma) = c\sigma$ is set, can be represented in the form

$$\Delta(\mu) = D(\mu) - cM(\mu) = 0, \tag{3}$$

where the degree of the polynomial $M(\mu)$ is always lower than the degree of the polynomial $D(\mu)$. In the case $n = 6$ we shall assume that

$$\sum_{\rho=1}^{6} \frac{\lambda_\rho M(\lambda_\rho)}{D'(\lambda_\rho)} = 0. \tag{4}$$

This, in particular, will be identically satisfied if the degree of the polynomial $M(\mu)$ is not higher than the third. For simplicity we assume that precisely this case occurs.

$$\text{* The case } n = 6,\ \lambda_n = 0,\ r = \sum_{s=1}^{6} j_s h_s = 0 \text{ reduces to } n = 5.$$

Thus, we assume that for $n = 5$

$$D(\mu) = -\mu^5 + k_1\mu^4 - k_2\mu^3 + k_3\mu^2 - k_4\mu + k_5; \tag{5}$$

$$M(\mu) = -l_1\mu^4 + l_2\mu^3 - l_3\mu^2 + l_4\mu - l_5; \tag{6}$$

for $n = 6$

$$D(\mu) = \mu^6 - m_1\mu^5 + m_2\mu^4 - m_3\mu^3 + m_4\mu^2 - m_5\mu + m_6; \tag{7}$$

$$M(\mu) = n_1\mu^3 - n_2\mu^2 + n_3\mu - n_4. \tag{8}$$

In accordance with the theorem of A. I. Lur' e, sufficient conditions of stability "in the large" for systems of the form (1) can be obtained as necessary and sufficient conditions for the existence of real solutions of a certain system of quadratic equations, called the resolving system.

In the cases under consideration, according to (^{1}), Chapter II, (8.16-21), taking (4) into account, the resolving system may be taken in the form:

for $n = 5$:

$$-(t_0\sigma_4)^2 + \Gamma_1 = 0,$$

$$(t_0\sigma_5 + t_1\sigma_4)^2 - 2t_0\sigma_4(t_0\sigma_6 + t_1\sigma_5 + t_2\sigma_4) + \Gamma_3 = 0,$$

$$\begin{aligned} &- (t_0\sigma_6 + t_1\sigma_5 + t_2\sigma_4)^2 - 2t_0\sigma_4(t_0\sigma_8 + t_1\sigma_7 + t_2\sigma_6 + t_3\sigma_5 + t_4\sigma_4) \\ &\quad + 2(t_0\sigma_5 + t_1\sigma_4)(t_0\sigma_7 + t_1\sigma_6 + t_2\sigma_5 + t_3\sigma_4) + \Gamma_5 = 0, \end{aligned} \tag{9}$$

$$-(t_4\sigma_{-1})^2+\Gamma_{-1}=0,$$

$$(t_4\sigma_{-2}+t_3\sigma_{-1})-2t_4\sigma_{-1}(t_4\sigma_{-3}+t_3\sigma_{-2}+t_2\sigma_{-1})+\Gamma_{-3}=0;$$

for $n=6$:

$$(t_1\sigma_5)^2+R_3=0,$$

$$-(t_1\sigma_6+t_2\sigma_5)^2+2t_1\sigma_5(t_1\sigma_7+t_2\sigma_6+t_3\sigma_5)+R_5=0,$$

$$-(t_5\sigma_{-1})^2+R_{-1}=0, \tag{10}$$

$$(t_5\sigma_{-2}+t_4\sigma_{-1})^2+2t_5\sigma_{-1}(t_5\sigma_{-3}+t_4\sigma_{-2}+t_3\sigma_{-1})+R_{-3}=0,$$

$$\begin{aligned}&-(t_5\sigma_{-3}+t_4\sigma_{-2}+t_3\sigma_{-1})^2-2t_5\sigma_{-1}(t_5\sigma_{-5}+t_4\sigma_{-4}+t_3\sigma_{-3}+t_2\sigma_{-2}+t_1\sigma_{-1})\\&\quad+2(t_5\sigma_{-2}+t_4\sigma_{-1})(t_5\sigma_{-4}+t_4\sigma_{-3}+t_3\sigma_{-2}+t_2\sigma_{-1})+R_{-5}=0.\end{aligned}$$

In systems (9) and (10):

$$t_i \text{ are unknowns;}$$

$$\Gamma_1=-l_1\sigma_5+l_2\sigma_4,$$

$$\Gamma_3=-l_1\sigma_7+l_2\sigma_6-l_3\sigma_5+l_4\sigma_4,$$

$$\Gamma_5=-l_1\sigma_9+l_2\sigma_8-l_3\sigma_7+l_4\sigma_6-l_5\sigma_5, \tag{11}$$

$$\Gamma_{-1}=-l_5\sigma_{-1},$$

$$\Gamma_{-3}=-l_3\sigma_{-1}+l_4\sigma_{-2}-l_5\sigma_{-3};$$

$$R_3=n_1\sigma_6-n_2\sigma_5,$$

$$R_5 = n_1\sigma_8 - n_2\sigma_7 + n_3\sigma_6 - n_4\sigma_5,$$

$$R_{-1} = -n_4\sigma_{-1}, \tag{12}$$

$$R_{-3} = -n_2\sigma_{-1} + n_3\sigma_{-1} - n_4\sigma_{-3},$$

$$R_{-5} = n_1\sigma_{-2} - n_2\sigma_{-3} + n_3\sigma_{-4} - n_4\sigma_{-5},$$

where it is assumed that

$$\Gamma_1 \neq 0, \qquad \Gamma_{-1} \neq 0, \qquad R_3 \neq 0, \qquad R_{-1} \neq 0,$$

since otherwise the problem reduces to the previously considered cases (^{1}).

The constants σ_k $(k = 1, 2, \ldots, n + 3)$, σ_{-k} $(k = 1, 2, 3, 4)$ are computed from the coefficients of the polynomials (5) and (7) according to (^{1}), Ch. II, (8.12-13), and the quantities σ_{-5} and σ_9, obtained by the method indicated in (^{1}), are equal to

$$\sigma_{-5} = \frac{m_2}{m_6^2} - \frac{m_4^2}{m_6^3} + \frac{3m_5m_3}{m_6^3} + \frac{3m_5^2m_4}{m_6^4} - \frac{m_5^4}{m_6^5}, \tag{13}$$

$$\sigma_{n+4} = \sigma_9 = -k_1^5 + 4k_1^3k_2 - 3k_1^2k_3 - 3k_1k_2^2 + 2k_2k_3 + 2k_1k_4 - k_5. \tag{14}$$

For the existence of real solutions of the resolvent systems (9) and (10), it is obviously necessary:

for $n = 5$

$$\Gamma_1 > 0, \qquad \Gamma_{-1} > 0; \tag{15}$$

for $n = 6$

$$R_3 < 0, \qquad R_{-1} > 0. \tag{16}$$

In this case, for system (9) we have

$$t_0 = \frac{\varepsilon\sqrt{\Gamma_1}}{\sigma_4}, \qquad t_4 = \frac{\varepsilon\sqrt{\Gamma_{-1}}}{\sigma_{-1}}, \qquad \varepsilon = \pm 1 \tag{17}$$

and for system (10)

$$t_1 = \frac{\varepsilon\sqrt{-R_3}}{\sigma_5}, \qquad t_5 = \frac{\varepsilon\sqrt{R_{-1}}}{\sigma_{-1}}, \qquad \varepsilon = \pm 1. \tag{18}$$

If (17) and (18) hold, then by transformations systems (9) and (10) can be brought to the same form:

$$\begin{aligned} x^2 - \varepsilon_1 z &= A, \\ y^2 - \varepsilon_2 z &= B, \qquad \varepsilon_{1,2} = \pm 1, \\ z^2 - xy &= C, \end{aligned} \tag{19}$$

where the following notation has been introduced:

for $n = 5$

$$x = \frac{t_1}{2\sqrt{2|t_0|\sqrt{|t_0t_4|}}}, \qquad y = \frac{t_3}{2\sqrt{2|t_4|\sqrt{|t_0t_4|}}}, \qquad z = \frac{t_2}{4\sqrt{|t_0t_4|}}, \tag{20}$$

$$A = \frac{t_0^2(2\sigma_4\sigma_6 - \sigma_5^2) - \Gamma_3}{8\sigma_4^2|t_0|\sqrt{|t_0t_4|}}, \qquad B = \frac{t_4^2(2\sigma_{-1}\sigma_{-3} - \sigma_{-2}^2) - \Gamma_{-3}}{8\sigma_{-1}^2|t_4|\sqrt{|t_0t_4|}},$$

$$C = \frac{-2t_0t_4\sigma_4^2 + 8A|t_0|\sqrt{|t_0t_4|}(2\sigma_4\sigma_6 - \sigma_5^2) - t_0^2(\sigma_6^2 + 2\sigma_4\sigma_8 - 2\sigma_5\sigma_7) + \Gamma_5}{16\sigma_4^2|t_0t_4|}; \tag{21}$$

for $n = 6$

$$x = \frac{t_2}{2\sqrt{2|t_1|\sqrt{|t_1t_5|}}}, \qquad y = \frac{t_4}{2\sqrt{2|t_5|\sqrt{|t_1t_5|}}}, \qquad z = \frac{t_2}{4\sqrt{|t_1t_5|}}, \tag{22}$$

$$A = \frac{t_1^2(2\sigma_5\sigma_7 - \sigma_6^2) + R_5}{8\sigma_5^2|t_1|\sqrt{|t_1t_5|}}, \qquad B = \frac{t_5^2(2\sigma_{-1}\sigma_{-3} - \sigma_{-2}^2) - R_{-3}}{8\sigma_{-1}^2|t_5|\sqrt{|t_1t_5|}}, \tag{23}$$

$$C = \frac{-2t_1t_5\sigma_{-1}^2 + 8B|t_5|\sqrt{|t_1t_5|}(2\sigma_{-1}\sigma_{-3} - \sigma_{-2}^2) - t_5^2(\sigma_{-3}^2 + 2\sigma_{-1}\sigma_{-5} - 2\sigma_{-2}\sigma_{-4}) + R_{-5}}{16\sigma_{-1}^2|t_1t_5|}.$$

(19) is an abbreviated notation for four different systems of quadratic equations. It is required to find conditions under which at least one of them has a real solution. It can be shown that

for the existence of a real solution of at least one of the systems (19), it is necessary and sufficient that the algebraic equation

$$z^4 - (2C+1)z^2 - (A+B)z + C^2 - AB = 0 \tag{24}$$

have at least one real root z_1 such that

$$z_1 \geqslant \max(-A, -B). \tag{25}$$

Obtaining the corresponding conditions with the aid of general criteria (^{2}) is difficult in the present case, owing to the unwieldiness of the relations obtained. Therefore it proved advisable to investigate directly the character of the change of the roots of equation (24) by means of special geometric constructions in the space of the parameters A, B, and C.

The final result of the consideration can be formulated as follows.

Theorem. *Let the inequalities (15) or (16) hold; then:*

a) *if*

$$C + \frac{1}{2} > 0, \qquad C + \frac{1}{4} + AB > 0, \tag{26}$$

$$D = 4\left[\frac{(2C+1)^2}{3} + 4C^2 - 4AB\right]^3 -$$

$$-27\left[\frac{2(2C+1)^3}{27} - \frac{8(2C+1)(C^2-AB)}{3} - (A+B)^2\right]^2 \geqslant 0, \tag{27}$$

then, for stability "in the large" of the unperturbed motion of the corresponding system (1), it is sufficient that at least one of the relations

$$A + B < 0, \qquad AB > C > \frac{1}{4}; \tag{28}$$

b) *if*

$$D < 0 \tag{29}$$

or

c) *if*

$$D = 0 \tag{30}$$

and (26) and the conditions

$$C + \frac{1}{2} < 0, \qquad C + \frac{1}{4} + AB = 0, \qquad A + B = 0, \tag{31}$$

are not satisfied, then, for stability it is sufficient that

$$A + B \geqslant 0. \tag{32}$$

The stability criteria obtained are the most general ones that can be obtained with the aid of A. I. Lur’ e’ s theorem for the systems under consideration.

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REFERENCES

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[2] N. N. Meĭman, *Uspekhi Mat. Nauk*, **4**, 136 (34) (1949).

Note: Figure translations are in progress. See original paper for figures.