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Abstract

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MATHEMATICS

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SOLUTION BY THE FOURIER METHOD OF NON-SELF-ADJOINT MIXED PROBLEMS FOR HYPERBOLIC SYSTEMS IN THE PLANE

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We shall solve the mixed problem for a hyperbolic system in the narrow sense [1]

$$\frac{\partial}{\partial t}u(x,t) = A(x)\frac{\partial}{\partial x}u(x,t) + B(x)u(x,t) \tag{1}$$

$(0 \leq x \leq l,\ 0 \leq t \leq T < +\infty)$ with boundary and initial conditions

$$\begin{aligned} &\text{a)} \quad M\frac{\partial}{\partial t}u(0,t) + Nu(0,t) + P\frac{\partial}{\partial t}u(l,t) + Qu(l,t) = 0; \\ &\text{b)} \quad u(x,0) = f(x), \end{aligned} \tag{2}$$

where $u(x,t)$ is an n-dimensional vector function with complex coordinates. The matrix $A(x)$ is twice, and $B(x)$ once, continuously differentiable for $x \in [0,l]$; moreover, the eigenvalues of the matrix $A(x)$ do not vanish on the segment $[0,l]$. The matrices M, N, P, Q are complex, and

$$\operatorname{rang}\begin{Vmatrix} M & P \\ N & Q \end{Vmatrix} = n + q, \qquad \operatorname{rang}\|M, P\| = q \quad (0 \leq q \leq n). \tag{3}$$

Let $D_2(0,l)$ be the Banach space of classes of measurable functions $f(x)$ $(0 \leq x \leq l)$, equivalent with respect to the norm

$$\|f(x)\|_{D_2(0,l)} = \left(\int_0^l \|f(x)\|^2\,dx + \|Mf(0) + Pf(l)\|^2\right)^{1/2},$$

and suppose that in condition (2b) $f(x) \in D_2(0,l)$.

Put in the boundary-value problem (1), (2a)

$$u(x,t) = y(x)e^{\lambda t}.$$

To find $y(x)$ $(0 \le x \le l)$ and λ we obtain the parametric problem

$$\begin{aligned} &\text{a)} \quad A(x)y'(x) + B(x)y(x) = \lambda y(x); \\ &\text{b)} \quad (M\lambda + N)y(0) + (P\lambda + Q)y(l) = 0. \end{aligned} \tag{4}$$

For system (4a) one can construct [2] a fundamental matrix $Y(x, \lambda)$, analytic in λ for each $x \in [0, l]$ in each of the regions: 1) $\operatorname{Re}\lambda < -\gamma$, 2) $|\operatorname{Re}\lambda| < \gamma$, 3) $\operatorname{Re}\lambda > \gamma$, where γ is a sufficiently large positive number, and having the asymptotic representation

$$Y(x, \lambda) = \left[K(x) + O\left(\frac{1}{\lambda}\right)\right] \exp\left[\lambda \int_0^x \Lambda(\xi)\, d\xi\right] \tag{5}$$

uniformly with respect to $x \in [0, l]$ as $\lambda \to \infty$. Here $\Lambda(x) = [\nu_1(x), \nu_2(x), \dots, \nu_n(x)]$ is the diagonal form of the matrix $A^{-1}(x)$:

$$\Lambda(x) = K^{-1}(x)A^{-1}(x)K(x),$$

and $K(x)$ may be chosen so that the functions $\nu_i(x)$ $(i = 1, 2, \dots, n;\ 0 \le x \le l)$ will be continuous and will be arranged in decreasing order–

order: $\nu_1(x) > \nu_2(x) > \cdots > \nu_m(x) > 0 > \nu_{m+1}(x) > \cdots > \nu_n(x)$. To find the eigenvalues of problem (4) we form the characteristic determinant $\Delta(\lambda) = \det[(M\lambda + N)Y(0, \lambda) + (P\lambda + Q)Y(0, l)]$ and expand it, using formula (5), as follows:

$$\Delta(\lambda) = \lambda^q \left[\varphi(\lambda) + \sum_{i=1}^{2^n} b_i(\lambda) e^{\alpha_i \lambda}\right],$$

where $b_i(\lambda) \to 0$ as $\lambda \to \infty$ $(i = 1, 2, \dots, 2^n)$;

$$\varphi(\lambda) = \sum_{i=1}^{2^n} a_i e^{\alpha_i \lambda}$$

is a certain Dirichlet polynomial, and the α_i $(i = 1, 2, \dots, 2^n)$ are the numbers arranged in decreasing order:

$$0, \quad \int_0^l \nu_i(\xi)\, d\xi$$

$$(i = 1, 2, \dots, 2^n),$$

the sums of these numbers taken two at a time, three at a time, and so on up to n; in particular,

$$\alpha_1 = \int_0^l \sum_{i=1}^m \nu_i(\xi)\, d\xi, \qquad \alpha_{2^n} = \int_0^l \sum_{i=m+1}^n \nu_i(\xi)\, d\xi.$$

Definition 1. The boundary conditions (4b) (and also (2a)) are called **regular** if $a_1 \neq 0,\ a_{2^n} \neq 0$.

Theorem 1. *If the boundary conditions are regular, then problem (4) has a countable set of eigenvalues, and all of them are located in the strip*

$$-\gamma < \operatorname{Re}\lambda < \gamma < +\infty.$$

Let $\{\lambda_s\}$ $(s = 0, \pm 1, \ldots)$ be the zeros of the function $\Delta(\lambda)$, renumbered in increasing order of their imaginary parts; let $\{k_s\}$ be their multiplicities; and let $\{p_s\}$ be the multiplicities of the eigenvalues λ_s; then, as is known ([3]), $p_s \leq k_s$. If $p_s < k_s$ for at least one s, then in the boundary-value problem (1), (2) we shall take the unknown function $u(x,t)$ to be a rectangular matrix, and set

$$u(x,t) = \mathcal{Y}_{nk}(x) e^{J_k(\lambda)t} \qquad (k = 1, 2, \ldots), \tag{6}$$

where $\mathcal{Y}_{nk}(x)$ $(0 \leq x \leq l)$ is an unknown rectangular matrix with n rows and k columns, and $J_k(\lambda)$ is a k-dimensional Jordan cell. To find $\mathcal{Y}_{nk}(x)$ $(k = 1, 2, \ldots)$ we obtain the parametric problems

$$\begin{aligned} &\text{a)}\quad A(x)\mathcal{Y}'_{nk}(x) + B(x)\mathcal{Y}_{nk}(x) = \mathcal{Y}_{nk}(x)J_k(\lambda);\\ &\text{b)}\quad M\mathcal{Y}_{nk}(0)J_k(\lambda) + N\mathcal{Y}_{nk}(0)J_k(\lambda) + P\mathcal{Y}_{nk}(l)J_k(\lambda) + Q\mathcal{Y}_{nk}(l) = 0. \end{aligned} \tag{7}$$

Using the theory of eigenfunctions and associated functions ([3,4]), we establish the following properties of these problems: 1) all problems (7), for $k = 1, 2, \ldots$, have common eigenvalues, namely the numbers λ_s $(s = 0, \pm 1, \ldots)$; 2) the number of linearly independent solutions of problem (7), as k increases, does not decrease and for $k \geq k_s$ is equal to k_s; 3) these k_s solutions can be chosen so that they consist of p_s groups of matrices:

$$\|0, \ldots, 0, y_1^{(i)}(x), y_2^{(i)}(x), \ldots, y_{m_i}^{(i)}(x)\|,$$

$$\|0, \ldots, 0, 0, y_1^{(i)}(x), \ldots, y_{m_i-1}^{(i)}(x)\|, \ldots, \|0, \ldots, 0, 0, 0, \ldots, 0, y_1^{(i)}(x)\|$$

$(i = 1, 2, \ldots, p_s)$, where $y_j^{(i)}(x)$ $(0 \leq x \leq l;\ i = 1, 2, \ldots, p_s;\ j = 1, 2, \ldots, m_i)$ are n-dimensional columns. From the matrices $\mathcal{Y}_{nm_i}^{(s)}(x) = \|y_1^{(i)}(x), y_2^{(i)}(x), \ldots, y_{m_i}^{(i)}(x)\|$ $(i = 1, 2, \ldots, p_s)$, by formula (6) we construct the matrices

$$U_{nm_i}^{(s)}(x,t) = \mathcal{Y}_{nm_i}^{(s)}(x) \exp J_{m_i}(\lambda_s)t \quad (s = 0, \pm 1, \ldots;\ i = 1, 2, \ldots, p_s),$$

and these latter, for each s $(s = 0, \pm 1, \ldots)$, are combined into one

$$U_{nk_s}^{(s)}(x,t) = \mathcal{Y}_{nk_s}^{(s)}(x) e^{I_s t}, \tag{8}$$

where

$$I_s = [J_{m_1}(\lambda_s),\ J_{m_2}(\lambda_s),\ \ldots,\ J_{m_{p_s}}(\lambda_s)]$$

is a quasidiagonal matrix;

$$y_{nk_s}^{(s)}(x) = \|y_{nm_1}^{(s)}(x),\ y_{nm_2}^{(s)}(x),\ \dots,\ y_{nm_{p_s}}^{(s)}(x)\|. \tag{9}$$

Each column of the matrix (8) is a solution of the boundary-value problem (1), (2a), and therefore, if $f(x) \in \overline{D}_2(0,l)$ is represented in the form of the sum of the series

$$f(x) = \sum_{s=-\infty}^{+\infty} y_{nk_s}^{(s)}(x) a_s, \tag{10}$$

then the formal solution of problem (1), (2) will have the form

$$\sum_{s=-\infty}^{+\infty} y_{nk_s}^{(s)}(x) e^{\lambda_s t} a_s. \tag{11}$$

Definition 2. The boundary conditions

$$s^* v(t) = -R\frac{\partial}{\partial t} v(0,t) + Sv(0,t) - V\frac{\partial}{\partial t} v(l,t) + Wv(l,t) = 0 \tag{12}$$

are called adjoint to the boundary conditions (2a) if there are matrices $\|H_1, H_2\|$ and $\|G_1, G_2\|$ such that, for arbitrary $u(x,t) \in C^{(1)}(\Omega)$, $v(x,t) \in C^{(1)}(\Omega)$ ($\Omega = [0,l] \times [0,T]$),

$$v^*(x,t)A(x)u(x,t)\Big|_{x=0}^{x=l} = \frac{\partial}{\partial t}[Rv(0,t) + Vv(l,t)]^*[Mu(0,t) + Pu(l,t)] +$$

$$+[H_1 v(0,t) + H_2 v(l,t)]^* su(t) + [s^* v(t)]^*[G_1 u(0,t) + G_2 u(l,t)], \tag{13}$$

where $su(t)$ is the left-hand side of the boundary condition (2a)*.

The matrices R, S, V, W are found by comparing the coefficients in the identity (13). Without restricting the generality of the results, in the condition (2a) one may assume

$$\|M, P, N, Q\| = \left\| \begin{matrix} M_{qn} & P_{qn} & N_{qn} & Q_{qn} \\ 0_{n-qn} & 0_{n-qn} & N_{n-qn} & Q_{n-qn} \end{matrix} \right\|,$$

where 0_{n-qn} is a matrix of zeros. Then it is not difficult to prove that

$$\|R, V, S, W\| = \left\| \begin{matrix} R_{qn} & V_{qn} & S_{qn} & W_{qn} \\ 0_{n-qn} & 0_{n-qn} & S_{n-qn} & W_{n-qn} \end{matrix} \right\|.$$

Definition 3. The boundary-value problem for the system

$$-\frac{\partial}{\partial t}v(x,t) = -\frac{\partial}{\partial x}[A^*(x)v(x,t)] + B^*(x)v(x,t) \tag{14}$$

with the boundary conditions (12) is called adjoint to problem (1), (2a).

Carrying out separation of variables in problem (14), (12) by the formula $v(x,t) = z(x)e^{-\mu t}$, we obtain the parametric problem:

$$\begin{aligned} &\text{a)} \quad -\frac{d}{dx}[A^*(x)z(x)] + B^*(x)z(x) = \mu z(x); \\ &\text{b)} \quad (R\mu + S)z(0) + (V\mu + W)z(l) = 0. \end{aligned} \tag{15}$$

Theorem 2. If λ_s is a zero of the function $\Delta(\lambda)$ of multiplicity k_s and an eigenvalue of problem (4) of multiplicity p_s, then $\mu_s = \overline{\lambda}_s$ will be a zero of the characteristic determinant $\Delta_1(\mu)$ of problem (15) of the same multiplicity k_s and an eigenvalue of problem (15) of the same multiplicity p_s.

Let $F_{nk}(x)$ and $G_{nm}(x)$ ($k, m = 1, 2, \dots;\ 0 \le x \le l$) be two rectangular matrices. Introduce the notation

$$[F_{nk}(x), G_{nm}(x)] =$$

$$= \int_0^l G^*_{nm}(x)F_{nk}(x)\,dx + [RG_{nm}(0) + VG_{nm}(l)]^*[MF_{nk}(0) + PF_{nk}(l)].$$

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* v^*u here and below denotes the scalar product of the vectors u and v.

Theorem 3. If $\mathcal{Y}^{(s)}_{nk_s}(x)$ $(s = 0, \pm 1, \dots)$ is the system of matrices constructed from the eigenfunctions and adjoint functions of problem (4), and $Z^{(r)}_{nk_r}(x)$ $(r = 0, \pm 1, \dots)$ is the same system for problem (15), then for $s \neq r$

$$[\mathcal{Y}^{(s)}_{nk_s}(x), Z^{(r)}_{nk_r}(x)] = 0.$$

Theorem 4. If $f(x) \in D_2(0,l)$ is represented in the form of the sum of the series (10), then

$$a_s = [f(x), B_s^{*-1} Z^{(s)}_{nk_s}(x)],$$

where

$$B_s = [\mathcal{Y}^{(s)}_{nk_s}(x), Z^{(s)}_{nk_s}(x)] \quad (s = 0, \pm 1, \dots).$$

To justify the scheme presented, we shall use the problems generated by (4) by the linear differential operator

$$\mathcal{L}y = A(x)y'(x) + B(x)y(x),$$

where $y(x) \in C^{(1)}(0,l)$ and

$$MA(0)y'(0) + [MB(0) + N]y(0) + PA(l)y'(l) + [PB(l) + Q]y(l) = 0.$$

Theorem 5. If the boundary conditions are regular, $f(x)$ $(0 \le x \le l)$ belongs to the domain of definition of the operator $\mathcal{L}y$, $f_1(x) = \mathcal{L}f(x)$ is continuous and satisfies the condition

$$N_{n-qn} f_1(0) + Q_{n-qn} f_1(l) = 0,$$

and $f_1'(x) \in \mathcal{L}_2(0,l)$, then the series (11), under a certain grouping of terms independent of the choice of the function $f(x)$, converges uniformly on Ω to a continuously differentiable function $u(x,t)$ satisfying equation (1) and conditions (2).

Introduce the Banach space $M_2(\Omega)$ of measurable functions $f(x,t)$, $(x,t) \in \Omega$, such that for each $t \in [0,T]$, $f(x,t)$ belongs to $D_2(0,l)$ and

$$\|f(x,t)\|_{D_2(0,l)} < m_0 < +\infty$$

uniformly with respect to t;

$$\|f(x,t)\|_{M_2\Omega} = \sup_{0 \le t \le T} \|f(x,t)\|_{D_2(0,l)}.$$

$M_2(\Omega)$ is a complete space.

Definition 4. A function $u(x,t) \in M_2(\Omega)$ is called a **generalized solution**[5] of problem (1), (2), if, for any $t \in [0,T]$ and any function $v(x,t) \in C^{(1)}(\Omega)$ for which

$$S_{n-qn} v(0,t) + W_{n-qn} v(l,t) = 0,$$

it satisfies the integral equation

$$\iint_{\Omega(t)} [\Phi^* v(x,\tau)]^* u(x,\tau)\,dx\,d\tau - [u(x,t), v(x,t)] + [f(x), v(x,0)] + \int_0^t [S^* v(\tau)]^* [Mu(0,\tau) + Pu(l,\tau)]\,d\tau = 0,$$

where

$$\Phi^* v(x,\tau) = \frac{\partial}{\partial \tau} v(x,\tau) - \frac{\partial}{\partial x}[A^*(x) v(x,\tau)] + B^*(x) v(x,\tau), \quad (x,\tau) \in \Omega(t) = [0,l] \times [0,t],$$

$$[u(x,t), v(x,t)] = \int_0^l v^*(x,t) u(x,t)\,dx + [Rv(0,t) + Vv(l,t)]^* [Mu(0,t) + Pu(l,t)].$$

Theorem 6. If the boundary conditions are regular and if $f(x) \in D_2(0,l)$, then the series (11) converges, under a certain grouping of its terms, in the norm of the space $M_2(\Omega)$ to a certain function $u(x,t)$, $(x,t) \in \Omega$, which is the unique generalized solution of problem (1), (2).

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CITED LITERATURE

1. I. G. Petrovskii, *Lectures on Partial Differential Equations*, 1950.

2. V. S. Pugachev, *Matem. sborn.*, **15**(57), 1, 24 (1944).

3. M. A. Naimark, *Linear Differential Operators*, 1954.

4. M. V. Keldysh, *DAN*, **72**, No. 1, 11 (1951).

5. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.

Note: Figure translations are in progress. See original paper for figures.