THE FIRST BOUNDARY-VALUE PROBLEM FOR QUASILINEAR ELLIPTIC EQUATIONS

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Abstract

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MATHEMATICS

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THE FIRST BOUNDARY-VALUE PROBLEM FOR QUASILINEAR ELLIPTIC EQUATIONS

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1. Consider, in a bounded domain Ω of Euclidean space $x = (x_1, \dots, x_n)$, the equation

$$Lu \equiv -\sum_{i,j=1}^{n} a_{ij}(x,u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x,u)\frac{\partial u}{\partial x_i} + a(x,u) = 0 \tag{1}$$

with the boundary condition

$$u\big|_S = 0, \tag{2}$$

where S is the boundary of the domain Ω. We shall assume that S consists of a finite number of twice differentiable surfaces with second derivatives satisfying a Hölder condition with a positive exponent.

An extensive literature is devoted to the first boundary-value problem for nonlinear elliptic equations (see, for example, ([1–5])). However, almost all results concern the case of an equation with two independent variables. This is due to the fact that the available methods for obtaining a priori estimates of solutions of nonlinear elliptic equations make essential use of the presence of only two spatial variables. For an arbitrary number of spatial variables, a priori estimates in C have been obtained, as far as we know, only for solutions of quasilinear parabolic equations of the form $du/dt + Lu = 0$ in the work of O. A. Ladyzhenskaya ([6]), on which we substantially rely. The methods of estimates given there are directly applicable also to equation (1).

Suppose further that $\partial a(x,u)/\partial u \geq \beta > 0$ for $x \in \overline{\Omega}$ and all u, and that the coefficients a_{ij}, a_i, a have first-order derivatives with respect to x and u and are bounded together with these derivatives by some constant C_4 for

$$x \in \overline{\Omega} \quad \text{and} \quad |u| \leq \frac{1}{\beta}\max_{x\in\overline{\Omega}} |a(x,0)| = C_1.$$

2. Theorem 1. *Suppose that for some* $\alpha = \text{const} > 0$ *and any real* ξ_i

$$\sum_{i,j=1}^{n} a_{ij}(x,u)\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2, \qquad \max\left|\frac{\partial a_{ij}}{\partial u}\right| \leq \frac{\alpha e\sqrt{3}}{12nC_1},$$

when $x \in \overline{\Omega}$, $|u| \leq C_1$. Then, under the conditions indicated in item 1, problem (1), (2) has a solution $u \in E_{\delta,2}$.[1]

To formulate the second theorem, suppose that for some $\alpha_0 = \text{const} > 0$ and arbitrary real ξ_i

$$\sum_{i,j=1}^{n} a_{ij}(x,0)\xi_i\xi_j \geq \alpha_0 \sum_{i=1}^{n} \xi_i^2,$$

where $x \in \overline{\Omega}$. Then it is easy to show that there exists a positive number α_1, and that the C_1 and C_4 indicated above can be chosen so that, for all $x \in \overline{\Omega}$ and $|u| \leq C_1 \leq \frac{e\sqrt{3}}{12n}\frac{\alpha_1}{C_4}$, the inequality

$$\sum_{i,j=1}^{n} a_{ij}(x,u)\xi_i\xi_j \geq \alpha_1 \sum_{i=1}^{n} \xi_i^2$$

holds.

Theorem 2. *Under the conditions indicated in item 1, problem (1), (2) has a solution $u \in E_{8,2}$, if*

$$\beta \geq \frac{12n}{e\sqrt{3}} \frac{C_4}{\alpha_1} \max_{x\in\overline{\Omega}} |a(x,0)|.$$

3. Both theorems are proved analogously. Replace equation (1) by an equation depending on a parameter $t \in [0,1]$:

$$L_{tu} \equiv -\sum_{i,j=1}^{n} a_{ij}(x,tu)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x,tu)\frac{\partial u}{\partial x_i} + \tilde{a}(x,t,u) = 0, \qquad (1')$$

where

$$\tilde{a}(x,t,u) = a(x,tu) + (1-t)\beta u.$$

[1] The space $E_{\delta,2}$ consists of functions u having second derivatives satisfying a Hölder condition with exponent δ, and

$$\|u\|^{\Omega}_{E_{\delta,2}} = \max_{\overline{\Omega}} |u| + \max_{\overline{\Omega}} |D_1 u| + \max_{\overline{\Omega}} |D_2 u| + \sup_{P,Q\in\overline{\Omega}} \frac{|D_2 u(P) - D_2 u(Q)|}{|P-Q|^{\delta}}.$$

Here max and sup are taken over all derivatives up to second order inclusive.

Suppose first that, in the domain indicated in item 1, the coefficients a_{ij}, a_i, a have first derivatives with respect to x and u, satisfying the Hölder condition with some positive exponent. The main point of the proof is the establishment of a priori estimates for the solutions $u_{(t)} = u(x,t)$ of problem $(1')$, (2), independent of t.

Lemma 1. *The solutions of problem $(1')$, (2) are bounded in modulus in the aggregate for all $t \in [0,1]$, namely*

$$|u_{(t)}| \leq \frac{1}{\beta} \max_{x \in \overline{\Omega}} |a(x,0)| = C_1.$$

The proof follows from the "maximum principle."

Lemma 2. *The moduli of the derivatives $\partial u_{(t)}/\partial x_i$ are bounded in the aggregate for all $t \in [0,1]$:*

$$|\operatorname{grad} u_{(t)}| \leq C_2,$$

where C_2 depends only on β, n, C_4, the domain Ω, and α or α_1. The proof is carried out separately for the boundary and inside the domain Ω and is a simple modification of the corresponding proof from work ([6]).

Lemma 3. *For the solution of problem $(1')$, (2), the estimate*

$$\|u_{(t)}\|_{E^{\Omega}_{8,2}} \leq C_3, \tag{3}$$

is valid for all $t \in [0,1]$, where C_3 depends on the same quantities as C_2.

The proof follows from Lemmas 1 and 2 and Schauder' s results ([7]) on linear elliptic equations.

Consider the linear elliptic equation

$$-\sum_{i,j=1}^{n} a_{ij}(x,tu) \frac{\partial^2 U}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x,tu) \frac{\partial U}{\partial x_i} + \tilde{a}(x,t,u) = 0$$

under the condition $U|_S = 0$, where u is some element of $E_{\delta,2}$. This equation induces an operator $A_{(t)}$, defined by the equality $A_{(t)}u = U_{(t)}$, which, acting in the space $E_{\delta,2}$, $0 < \delta < 1$, will be completely continuous and uniformly continuous in t. By virtue of the assumptions on the coefficients a_{ij}, a_i, a, made at the beginning of item 3, $U_{(t)}$ will be three times continuously differentiable inside the domain Ω.

By Lemma 3, for any solution $u_{(t)} \in E_{\delta,2}$ of the functional equation

$$u - A_{(t)} u = 0$$

the inequality (3) holds. For $t = 0$ the functional equation (4) is equivalent to the linear elliptic equation into which equation $(1')$ is transformed if one sets $t = 0$ in it. It follows from (7) that this linear elliptic equation has a unique solution $u_{(0)} \in E_{\delta,2}$. Hence it follows that for $t = 0$ the total index of the solutions (in the sense of (3^1)) of equation (4) is different from zero. From the results of Leray and Schauder ([3]) it follows that it is different from zero for all $t \in [0,1]$ and, consequently, for all t in the indicated interval (4) has at least one solution $u \in E_{\delta,2}$, for which the estimate (3) is valid. But for $t = 1$ equation (4) is equivalent to equation (1).

The restriction imposed in the proof on the coefficients was needed by us in order that the solution be three times continuously differentiable inside the domain. This condition was required in the proof of Lemma 2.

Let now the coefficients of equation (1) possess the properties indicated in item 1. We approximate them by functions, uniformly bounded together with their first derivatives, which satisfy the Hölder condition with some positive exponents. For the approximating equation, as has just been shown, problem (1), (2) has a solution. Passing to the limit, we prove the existence of a solution of the original problem.

4. For sufficiently large β, by the classical device indicated by Hadamard, one can prove uniqueness of the solution of problem (1), (2). Let $u \in E_{\delta,2}$ be some solution of problem (1), (2). By Lemma 1, $|u| \leq C_1$, and by Lemma 3, $|D_1 u| + |D_2 u| \leq C_3$.

Theorem 3. *If the conditions of Theorem 1 or 2 are fulfilled and $\beta > C_3 C_4$, then the solution of problem (1), (2) is unique.*

5. By the very same methods one can prove analogous theorems for quasilinear elliptic equations of a more general form

$$-\sum_{i,j=1}^{n} a_{ij}(x,u) \frac{\partial^2 u}{\partial x_i \partial x_j} + b(x,u,p) = 0,$$

where $p = (u_{x_1}, \dots, u_{x_n})$. In this case the same conditions as above are imposed on the coefficients a_{ij}, while for the coefficient b it is sufficient to require that $\partial b(x,u,p)/\partial u \geq \beta$ for $x \in \overline{\Omega}$ and all u and p, and, in addition, that the coefficient b, together with its first derivatives with respect to all arguments, be bounded by the constant C_4 for $x \in \overline{\Omega}$, $|u| \leq \frac{1}{\beta} \max_{x \in \overline{\Omega}} |b(x,0,0)| = C_1$, and all p.

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CITED LITERATURE

1. S. Bernstein, *Math. Ann.*, **69**, No. 1, 82 (1910).

2. J. Schauder, *Math. Zs.*, **37**, No. 4, 623 (1933).

3. J. Leray, J. Schauder, *Ann. de l' Ecole Norm. sup.*, **51**, No. 3, 45 (1934).

4. L. Nirenberg, *Comm. Pure and Appl. Math.*, **6**, No. 1, 103 (1953).

5. I. N. Vekua, *DAN*, **101**, No. 4, 593 (1956).

6. O. A. Ladyzhenskaya, *DAN*, **107**, No. 5, 636 (1956).

7. J. Schauder, *Math. Zs.*, **38**, No. 2, 257 (1934).

Note: Figure translations are in progress. See original paper for figures.