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Abstract

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AERODYNAMICS

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EQUATIONS OF THE THREE-DIMENSIONAL BOUNDARY LAYER IN A COMPRESSIBLE GAS FOR AN ARBITRARY SURFACE

(Presented by Academician A. A. Dorodnitsyn, 4 XII 1956)

In the author's work ⁽¹⁾ a rigorous derivation was given of the equations of the spatial boundary layer for an arbitrary surface in a viscous incompressible fluid, and certain transformations were indicated which simplify the structure of the three-dimensional equations and reveal the influence of the principal curvatures of the surface.

In the present paper a rigorous derivation is given of the equations of the three-dimensional boundary layer in a viscous compressible gas for an arbitrary surface, and transformations are indicated which bring these equations to a form analogous to the equations of the three-dimensional layer in a viscous incompressible fluid.

Let us write the equations of motion of a viscous compressible gas in dimensionless quantities. As scales we shall use the values of the velocity, density, pressure, absolute temperature, coefficient of viscosity, and thermal conductivity at a certain point of the flow; as the length scale we shall take a certain characteristic size of the body being flowed around. In this case the complete system of equations of motion of a viscous gas in vector form may be written as

$$\bar{\rho} \left(\text{grad} \frac{\bar{V}^2}{2} + [\text{rot} \bar{V} \cdot \bar{V}] \right) + \frac{1}{kM^2} \text{grad} \bar{P} = -\frac{2}{3\text{Re}} \text{grad} \bar{\mu} \text{div} \bar{V} + \frac{1}{\text{Re}} \text{div} \bar{\Pi}; \quad (1)$$

$$\text{div} \bar{\rho} \bar{V} = 0; \quad (2)$$

$$\bar{\rho} \left(\bar{V} \cdot \text{grad} \left[\bar{T} + \frac{k-1}{2} M^2 V^2 \right] \right) =$$

$$= \frac{1}{\text{Pr Re}} \text{div} \bar{\lambda} \text{grad} \bar{T} - \frac{2}{3} (k-1) \frac{M^2}{\text{Re}} \text{div} (\bar{\mu} \text{div} \bar{V} \cdot \bar{V}) + \frac{k-1}{\text{Re}} M^2 \text{div} (\bar{\Pi} \bar{V}); \quad (3)$$

$$\bar{P} = \bar{\rho}\bar{T}. \quad (4)$$

Here the following notation has been adopted: $\bar{\Pi}$ is the stress tensor with components:

$$\begin{aligned} \bar{\tau}_{xx} &= 2\bar{\mu}\frac{\partial\bar{u}}{\partial x}; & \bar{\tau}_{yy} &= 2\bar{\mu}\frac{\partial\bar{v}}{\partial y}; & \bar{\tau}_{zz} &= 2\bar{\mu}\frac{\partial\bar{w}}{\partial z}; \\ \bar{\tau}_{xy} = \bar{\tau}_{yx} &= \bar{\mu}\left(\frac{\partial\bar{u}}{\partial y} + \frac{\partial\bar{v}}{\partial x}\right); & \bar{\tau}_{xz} = \bar{\tau}_{zx} &= \bar{\mu}\left(\frac{\partial\bar{u}}{\partial z} + \frac{\partial\bar{w}}{\partial x}\right); & \bar{\tau}_{yz} = \bar{\tau}_{zy} &= \bar{\mu}\left(\frac{\partial\bar{v}}{\partial z} + \frac{\partial\bar{w}}{\partial y}\right); \end{aligned}$$

μ and λ are, respectively, the coefficients of viscosity and thermal conductivity; $k = c_p/c_v$.

On the surface of the body being flowed around the boundary conditions $\bar{u} = \bar{v} = \bar{w} = 0$ must be satisfied and, for steady flow, the equality

heat fluxes received and given off by the body. At infinity the velocities and temperature must take prescribed values.

To derive the boundary-layer equations near an arbitrary surface, we choose the following system of curvilinear orthogonal coordinates. On the surface under consideration we take the system of lines of curvature $\xi = \text{const}$ and $\eta = \text{const}$ as the system of curvilinear coordinates of the surface. Then the surfaces $\zeta = \text{const}$, parallel to the surface S , together with the surfaces $\xi = \text{const}$ and $\eta = \text{const}$ formed by the normals to the corresponding lines of curvature, form a triorthogonal system of surfaces. We shall take this system of surfaces as the new system of curvilinear orthogonal coordinates. As was shown in [1], for this coordinate system we shall have:

$$dS^2 = A_1^2 \left(1 + \frac{\zeta}{R_1}\right)^2 d\xi^2 + A_2^2 \left(1 + \frac{\zeta}{R_2}\right)^2 d\eta^2 + d\zeta^2,$$

where $A_1(\xi, \eta)$, $A_2(\xi, \eta)$ are the coefficients of the first fundamental form of the surface under consideration; $R_1(\xi, \eta)$, $R_2(\xi, \eta)$ are the principal radii of curvature of this surface.

For the chosen coordinate system the Lamé coefficients are equal to

$$H_1(\xi, \eta, \zeta) = A_1 \left(1 + \frac{\zeta}{R_1}\right); \quad H_2(\xi, \eta, \zeta) = A_2 \left(1 + \frac{\zeta}{R_2}\right); \quad H_3(\xi, \eta, \zeta) = 1.$$

Using the known expressions for the operations grad, div, rot in an arbitrary curvilinear orthogonal coordinate system and taking into account the following

expressions for the components of the stress tensor and the divergence of the stress tensor:

$$\bar{T}_{ii} = 2\mu \left(\frac{\partial}{\partial x^i} \left(\frac{\bar{v}_i}{H_i} \right) + \sum_{k=1}^3 \frac{\bar{v}_k}{H_i H^k} \frac{\partial H_i}{\partial x^k} \right); \quad \bar{T}_{ik} = \mu \left\{ \frac{H_i}{H_k} \frac{\partial}{\partial x^k} \frac{\bar{v}_i}{H_i} + \frac{H_k}{H_i} \frac{\partial}{\partial x^i} \frac{\bar{v}_k}{H_k} \right\};$$

$$(\operatorname{div} \bar{\Pi})_i = \frac{1}{H_i} \left\{ \frac{1}{H_1 H_2 H_3} \sum_{k=1}^3 \frac{\partial}{\partial x^k} \left(\frac{H_1 H_2 H_3 H_i}{H_k} \bar{T}_{ki} \right) - \sum_{k=1}^3 \frac{\bar{T}_{kk}}{H_k} \frac{\partial H_k}{\partial x^i} \right\},$$

one can write equations (1)–(3) in curvilinear coordinates ξ, η, ζ .

Next, in these equations we pass to the new variables

$$\xi = \xi_1, \quad \eta = \eta_1, \quad \zeta = \frac{\zeta_1}{\sqrt{\operatorname{Re}}}; \quad \bar{u} = u_1, \quad \bar{v} = v_1, \quad \bar{w} = \frac{w_1}{\sqrt{\operatorname{Re}}}; \quad \bar{T} = T_1.$$

Assume that in the new variables the quantities u_1, v_1, w_1, T_1 , as well as their first and second derivatives, have finite values, and that the surface of the body being flowed around has a smooth form with continuous principal curvatures and their derivatives. In this case expressions of the form $\frac{1}{R_1 \sqrt{\operatorname{Re}}}$, $\frac{1}{R_2 \sqrt{\operatorname{Re}}}$, $\frac{1}{R_1 \sqrt{\operatorname{Re}}} \frac{1}{R_1} \frac{\partial R_1}{\partial \xi_1}$, and so forth will tend to zero as the Reynolds number Re increases without bound.

Carrying out the transformations indicated above, from equations (1)–(4), as the Reynolds number increases without bound, we obtain the following system of equations of the three-dimensional boundary layer in a compressible gas for an arbitrary surface:

$$\rho \left(\frac{u}{A_1} \frac{\partial u}{\partial \xi} + \frac{v}{A_2} \frac{\partial u}{\partial \eta} + w \frac{\partial u}{\partial \zeta} + \frac{uw}{A_1 A_2} \frac{\partial A_1}{\partial \eta} - \frac{v^2}{A_1 A_2} \frac{\partial A_2}{\partial \xi} \right) = -\frac{1}{A_1} \frac{\partial P}{\partial \xi} + \frac{\partial}{\partial \zeta} \mu \frac{\partial u}{\partial \zeta}; \quad (5)$$

$$\rho \left(\frac{u}{A_1} \frac{\partial v}{\partial \xi} + \frac{v}{A_2} \frac{\partial v}{\partial \eta} + w \frac{\partial v}{\partial \zeta} + \frac{uv}{A_1 A_2} \frac{\partial A_2}{\partial \xi} - \frac{u^2}{A_1 A_2} \frac{\partial A_1}{\partial \eta} \right) = -\frac{1}{A_2} \frac{\partial P}{\partial \eta} + \frac{\partial}{\partial \zeta} \mu \frac{\partial v}{\partial \zeta}; \quad (6)$$

$$\rho \left(\frac{u}{A_1} \frac{\partial}{\partial \xi} + \frac{v}{A_2} \frac{\partial}{\partial \eta} + w \frac{\partial}{\partial \zeta} \right) \left(J c_{pT} + \frac{u^2 + v^2}{2} \right) = J \frac{\partial}{\partial \zeta} \lambda \frac{\partial T}{\partial \zeta} + \frac{\partial}{\partial \zeta} \mu \frac{\partial}{\partial \zeta} \frac{u^2 + v^2}{2}; \quad (7)$$

$$\frac{1}{A_1} \frac{\partial \rho u}{\partial \xi} + \frac{1}{A_2} \frac{\partial \rho v}{\partial \eta} + \frac{\partial \rho w}{\partial \zeta} + \frac{\rho u}{A_1 A_2} \frac{\partial A_2}{\partial \xi} + \frac{\rho v}{A_1 A_2} \frac{\partial A_1}{\partial \eta} = 0; \quad (8)$$

$$P = \rho RT. \quad (9)$$

On a curved surface the pressure gradient in the normal direction is a quantity of first order,

$$\frac{\partial P}{\partial \zeta} = \rho \left(\frac{u^2}{R_1} + \frac{v^2}{R_2} \right).$$

However, over the extent of the thin boundary layer it will be a small quantity

$$\left(\frac{\partial P_1}{\partial \zeta_1} = 0 \right),$$

and it should be neglected. Thus, the pressure, as usual, will be determined by the external flow. Therefore the boundary conditions for this system of equations will be:

$$\text{for } \zeta = 0 \quad u = v = w = 0$$

and the conditions of equality of heat fluxes:

$$\text{for } \zeta \rightarrow \infty \quad u = U_0(\xi, \eta), \quad v = V_0(\xi, \eta), \quad T = T_0(\xi, \eta),$$

where U_0 , V_0 , and T_0 are prescribed functions.

The obtained system of equations (5)–(9), with the boundary conditions, makes it possible, for an arbitrary surface, to investigate gas flow in a three-dimensional boundary layer at large supersonic velocities and high stagnation temperatures. As applied to particular classes of surfaces, this system of equations can be transformed to a simpler form (see (1)). However, in the general case the equations of a three-dimensional boundary layer in a compressible gas can be transformed to a form analogous to the equations of a three-dimensional layer in a viscous incompressible fluid.

First introduce new unknown functions \bar{u} , \bar{v} , and \bar{w} :

$$A_2 u = \omega \bar{u}, \quad A_1 v = \omega \bar{v}, \quad A_1 A_2 w = \omega \bar{w}.$$

The quantity ω may, for example, be set equal to

$$\omega = 1/A_1 A_2.$$

Transforming the equations to the new unknown functions, we find:

$$\rho\omega^2 \left[\bar{u} \frac{\partial \bar{u}}{\partial \xi} + \bar{v} \frac{\partial \bar{u}}{\partial \eta} + \bar{w} \frac{\partial \bar{u}}{\partial \zeta} - \bar{u}^2 \frac{\partial}{\partial \xi} \ln A_1 A_2^2 - 2\bar{u}\bar{v} \frac{\partial}{\partial \eta} \ln A_2 - \bar{v}^2 \left(\frac{A_2}{A_1} \right)^2 \frac{\partial}{\partial \xi} \ln A_2 \right] = -\frac{1}{A_1^2 \omega^2} \frac{\partial P}{\partial \xi} + \frac{\partial}{\partial \zeta} \mu \frac{\partial \bar{u}}{\partial \zeta};$$

$$\rho\omega^2 \left[\bar{u} \frac{\partial \bar{v}}{\partial \xi} + \bar{v} \frac{\partial \bar{v}}{\partial \eta} + \bar{w} \frac{\partial \bar{v}}{\partial \zeta} - \bar{v}^2 \frac{\partial}{\partial \eta} \ln A_1^2 A_2 - 2\bar{u}\bar{v} \frac{\partial}{\partial \xi} \ln A_1 - \bar{u}^2 \left(\frac{A_1}{A_2} \right)^2 \frac{\partial}{\partial \eta} \ln A_1 \right] = -\frac{1}{A_2^2 \omega^2} \frac{\partial P}{\partial \eta} + \frac{\partial}{\partial \zeta} \mu \frac{\partial \bar{v}}{\partial \zeta};$$

$$\frac{\partial \rho\omega \bar{u}}{\partial \xi} + \frac{\partial \rho\omega \bar{v}}{\partial \eta} + \frac{\partial \rho\omega \bar{w}}{\partial \zeta} = 0;$$

$$\rho\omega^2 \left(\bar{u} \frac{\partial}{\partial \xi} + \bar{v} \frac{\partial}{\partial \eta} + \bar{w} \frac{\partial}{\partial \zeta} \right) T^* = \frac{\partial}{\partial \zeta} \mu \frac{\partial T^*}{\partial \zeta} + \left(\frac{1}{Pr} - 1 \right) \frac{\partial}{\partial \zeta} \mu \frac{\partial}{\partial \zeta} J_{c_{pT}},$$

where

$$T^* = J_{c_{pT}} + \frac{u^2 + v^2}{2}.$$

We now pass to new independent variables

$$\xi_1 = \xi, \quad \eta_1 = \eta, \quad \xi_1 = \int_0^\zeta \rho\omega d\zeta.$$

These transformations generalize the known transformations applied by A. A. Dorodnitsyn to the two-dimensional boundary layer (2).

If we now introduce a new unknown function

$$\tilde{w} = u \frac{\partial \zeta_1}{\partial \xi} + \bar{v} \frac{\partial \zeta_1}{\partial \eta} + \bar{w} \frac{\partial \zeta_1}{\partial \zeta},$$

then the equations of the three-dimensional boundary layer in a gas take the form

$$\bar{u} \frac{\partial \bar{u}}{\partial \xi_1} + \bar{v} \frac{\partial \bar{u}}{\partial \eta_1} + \tilde{w} \frac{\partial \bar{u}}{\partial \zeta_1} - \bar{u}^2 \frac{\partial}{\partial \xi_1} \ln A_1 A_2^2 - 2\bar{u}\bar{v} \frac{\partial}{\partial \eta_1} \ln A_2 - \bar{v}^2 \left(\frac{A_2}{A_1} \right)^2 \frac{\partial}{\partial \xi_1} \ln A_2 =$$

$$= -\frac{1}{\rho A_1^2 \omega^4} \frac{\partial P}{\partial \xi_1} + \frac{\partial}{\partial \zeta_1} \rho \mu \frac{\partial \bar{u}}{\partial \zeta_1};$$

$$\bar{u} \frac{\partial \bar{v}}{\partial \xi_1} + \bar{v} \frac{\partial \bar{v}}{\partial \eta_1} + \tilde{w} \frac{\partial \bar{v}}{\partial \zeta_1} - \bar{v}^2 \frac{\partial}{\partial \eta_1} \ln A_1^2 A_2 - 2\bar{v} \tilde{w} \frac{\partial}{\partial \xi_1} \ln A_1 - \bar{u}^2 \left(\frac{A_1}{A_2} \right)^2 \frac{\partial}{\partial \eta_1} \ln A_1 =$$

$$= -\frac{1}{\rho A_2^2 \omega^4} \frac{\partial P}{\partial \eta_1} + \frac{\partial}{\partial \zeta_1} \rho \mu \frac{\partial \bar{v}}{\partial \zeta_1};$$

$$\frac{\partial \bar{u}}{\partial \xi_1} + \frac{\partial \bar{v}}{\partial \eta_1} + \frac{\partial \tilde{w}}{\partial \zeta_1} = 0; \quad P = \rho RT;$$

$$\bar{u} \frac{\partial T^*}{\partial \xi_1} + \bar{v} \frac{\partial T^*}{\partial \eta_1} + \tilde{w} \frac{\partial T^*}{\partial \zeta_1} = \frac{\partial}{\partial \zeta_1} \rho \mu \frac{\partial T^*}{\partial \zeta_1} + \left(\frac{1}{\text{Pr}} - 1 \right) \frac{\partial}{\partial \zeta_1} \rho \mu \frac{\partial J c_p T}{\partial \zeta_1}.$$

The boundary conditions for this system may be written in the form

$$\text{at } \zeta = 0 \quad \bar{u} = \bar{v} = \tilde{w} = 0$$

and the condition of equality of heat fluxes.

$$\text{at } \zeta = \infty \quad \bar{u} = A_1 A_2^2 U_0(\xi, \eta); \quad \bar{v} = A_1^2 A_2 V_0(\xi, \eta); \quad T = T_0(\xi, \eta).$$

This system of equations makes it possible to investigate the thermal boundary layer in a compressible gas near an arbitrary surface. As is seen, the principal inertial terms in the equations of motion and the equation of continuity have been transformed to a form analogous to the corresponding equations for an incompressible fluid over a flat plate. I note that the character of the gas flow in the boundary layer on a curved surface is affected by what is effectively the density

$$\rho_{\text{eff}} = \rho \omega = \rho / A_1 A_2.$$

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References

1. V. V. Struminskii, *DAN*, **108**, No. 4 (1956).

2. A. A. Dorodnitsyn, *Prikl. matem. i mekh.*, **6**, 6 (1942).

Note: Figure translations are in progress. See original paper for figures.

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