



Soviet-era science, translated into English

Mathematics

Z. L. Leibenzon

1957

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-195701.60903>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Mathematics

Z. L. Leibenzon

On the Estimation of Eigenvalues of Self-Adjoint Operators*

(Presented by Academician S. L. Sobolev on 31 V 1957)

1°. Let A be a self-adjoint (not necessarily bounded) operator in a Hilbert space H . Suppose that the operator A has a discrete spectrum.

Take an element $x \in H$; then

$$x = \sum_i x_i e_i,$$

where all $x_i \neq 0$, and the normalized eigenfunctions e_1, e_2, \dots of the operator A correspond to distinct eigenvalues $\lambda_1, \lambda_2, \dots$

In the totality M of functions $F(\lambda)$ for which the series

$$\sum_i |F(\lambda_i)|^2 |x_i|^2 = \|F\|^2$$

converges, introduce the scalar product

$$(F, G) = \sum_i F(\lambda_i) \overline{G(\lambda_i)} |x_i|^2. \quad (1)$$

Lemma. Let $F(\lambda)$ and $\Phi(\lambda)$ be functions from M and $\|\Phi\| > 0$. Then there exists an eigenvalue λ of the operator A such that $\Phi(\lambda) \neq 0$ and

$$\frac{|F(\lambda)|}{|\Phi(\lambda)|} \leq \frac{\|F\|}{\|\Phi\|}.$$

Proof. Since

$$\sum_i |\Phi(\lambda_i)|^2 |x_i|^2 = \|\Phi\|^2 > 0,$$

we have $\Phi(\lambda_i) \neq 0$ for some λ_i . As is easy to see, either

$$\frac{\|F\|^2}{\|\Phi\|^2} = \frac{\sum_i |F(\lambda_i)|^2 |x_i|^2}{\sum_i |\Phi(\lambda_i)|^2 |x_i|^2} > \inf_{i, \Phi(\lambda_i) \neq 0} \frac{|F(\lambda_i)|^2}{|\Phi(\lambda_i)|^2},$$

or

$$\frac{|F(\lambda_i)|}{|\Phi(\lambda_i)|} = \frac{\|F\|}{\|\Phi\|},$$

if $\Phi(\lambda_i) \neq 0$. In both cases the lemma is true.

Suppose it is known that $A^n x \in H$. Then M contains the subspace M_n of polynomials in λ of degree not exceeding n . The scalar products of polynomials in λ of degree not exceeding n , and in particular their norms, are easily compu-

* Reported at the seminar on differential equations at the Institute of Mathematics of the Academy of Sciences of the Ukrainian SSR in 1950.

compute, knowing the numbers

$$p_{\nu+\mu} = (\lambda^\nu, \lambda^\mu) = \sum_i \lambda_i^{\nu+\mu} |x_i|^2 = (A^\nu x, A^\mu x), \quad 0 \leq \nu, \mu \leq n. \quad (2)$$

Applying the lemma, taking in it as $F(\lambda)$ and $\Phi(\lambda)$ polynomials of degree not exceeding n , differing by a linear factor, we shall obtain intervals containing some eigenvalues of the operator A .

Theorem 1. *Let*

$$\varepsilon = \frac{\|(a_0 + a_1 \lambda + \dots + a_{n-1} \lambda^{n-1})(\lambda - \alpha)\|}{\|a_0 + a_1 \lambda + \dots + a_{n-1} \lambda^{n-1}\|} = \frac{\left[\sum_{\nu, \mu=0}^{n-1} (p_{\nu+\mu+2} - 2\alpha p_{\nu+\mu+1} + \alpha^2 p_{\nu+\mu}) a_\nu \bar{a}_\mu \right]^{1/2}}{\left[\sum_{\nu, \mu=0}^{n-1} p_{\nu+\mu} a_\nu \bar{a}_\mu \right]^{1/2}},$$

where a_0, a_1, \dots, a_{n-1} are arbitrary complex numbers; α is an arbitrary real number.

Then the interval $[\alpha - \varepsilon, \alpha + \varepsilon]$ contains one of the eigenvalues of the operator A .

If the elements $A^\nu x \in H$ are known for $0 \leq \nu \leq n$, then one can compute the numbers $p_{\nu+\mu}$, equal to their scalar products $(A^\nu x, A^\mu x)$ in the Hilbert space H ; then it is easy to compute the interval $[\alpha - \varepsilon, \alpha + \varepsilon]$. Putting $n = 1$ in Theorem 1, we obtain that for any real value α the interval

$$\left[\alpha - \sqrt{\frac{p_2 - 2\alpha p_1 + \alpha^2 p_0}{p_0}}, \quad \alpha + \sqrt{\frac{p_2 - 2\alpha p_1 + \alpha^2 p_0}{p_0}} \right]$$

contains some eigenvalue of the operator A .

For a self-adjoint differential operator A , this assertion for $\alpha = p_1/p_0$ coincides with the result of N. N. Bogolyubov and N. M. Krylov ⁽¹⁾.

2°. In Theorem 1 the length 2ε of the interval containing one of the eigenvalues is the ratio of the norms of polynomials whose degrees differ by 1. Therefore, in order to make ε as small as possible, we use polynomials of least norm. Namely, from all polynomials of some degree m ($m \leq n$) with leading coefficient 1, choose the polynomial $D_m(\lambda)$ with the least norm $\|D_m\|$.

Let $D_m(r) = 0$, so that $D_m(\lambda) = (\lambda - r)\varphi(\lambda)$. Here $\varphi(\lambda)$ is a polynomial of degree $m - 1$ with leading coefficient 1, and therefore $\|\varphi\| \geq \|D_{m-1}\|$. Let in the lemma $F(\lambda) = D_m(\lambda)$ and $\Phi(\lambda) = \varphi(\lambda)$. On the basis of the lemma there exists an eigenvalue λ of the operator A such that

$$|\lambda - r| \leq \frac{\|D_m\|}{\|\varphi\|} \leq \frac{\|D_m\|}{\|D_{m-1}\|}.$$

For the scalar product (1), the orthogonal polynomials of the theory of moments ⁽²⁾ are the polynomials $D_0(\lambda), D_1(\lambda), \dots, D_n(\lambda)$; in the theory of moments ⁽²⁾ they are studied.

We have the formulas

$$D_m(\lambda) = \frac{\begin{vmatrix} p_0 & \cdots & p_{m-1} & 1 \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ p_m & \cdots & p_{2m-1} & \lambda^m \end{vmatrix}}{\begin{vmatrix} p_0 & \cdots & p_{m-1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ p_{m-1} & \cdots & p_{2m-2} \end{vmatrix}}; \quad \|D_m\|^2 = \frac{\begin{vmatrix} p_0 & \cdots & p_m \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ p_m & \cdots & p_{2m} \end{vmatrix}}{\begin{vmatrix} p_0 & \cdots & p_{m-1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ p_{m-1} & \cdots & p_{2m-2} \end{vmatrix}}. \quad (3)$$

The computation of the polynomials $D_m(\lambda)$ and the numbers $\|D_m\|$ is much easier to carry out not by formulas (3), but by using the orthogonality property. For example, we have the recurrence formulas (3)

$$\|D_{m-1}\|^2 = (\lambda^{m-1}, D_{m-1}), \quad a_m = \frac{(\lambda^m, D_{m-1})}{\|D_{m-1}\|^2},$$

$$D_m(\lambda) = \lambda D_{m-1}(\lambda) + (a_{m-1} - a_m) D_{m-1}(\lambda) - \frac{\|D_{m-1}\|^2}{\|D_{m-2}\|^2} D_{m-2}(\lambda).$$

The polynomial $D_m(\lambda)$ has m distinct real roots ⁽²⁾. From what has been said it follows:

Theorem 2. *Let r_1, \dots, r_m be the m distinct real roots of the polynomial $D_m(\lambda)$, defined by formula (3).*

There exist eigenvalues $\lambda^{(1)}, \dots, \lambda^{(m)}$ of the operator A such that

$$|\lambda^{(k)} - r_k| \leq \frac{\|D_m\|}{\|D_m(\lambda)/(\lambda - r_k)\|} \leq \frac{\|D_m\|}{\|D_{m-1}\|}, \quad k = 1, \dots, m.$$

3°. It is useful to be able to choose the initial element x close to an eigenfunction of the operator A or to a linear combination of eigenfunctions. Then the intervals obtained in 1° and 2° will give narrower bounds for the corresponding eigenvalues.

We can always choose the initial element close to a linear combination of eigenfunctions corresponding to the largest eigenvalues of the operator A , taking the iterated element $A^t x$ instead of x .

Let, as the initial element, instead of x , the element $y = A^t x$ be taken, where $t \geq 0$ is an integer. Then the norm $\|F\|$, the scalar product (F, G) , the numbers p_k , and the polynomials $D_m(\lambda)$ are replaced, respectively, by the norm $\|F\|_t$, the scalar product $(F, G)_t$, the numbers q_k , and the polynomials $D_{m,t}(\lambda)$. It is easy to see that $q_k = p_{k+2t}$.

If A is a bounded operator, then for any $n \geq 0$, $A^n x \in H$, and the numbers $q_m = p_{m+2t}$ and the polynomials $D_{m,t}(\lambda)$ are defined by us for all $t, m \geq 0$.

Theorem 3. *Suppose that for some m*

$$\gamma = \sup_{j>m} |\lambda_j| < \min_{i=1, \dots, m} |\lambda_i|.$$

Then the roots $r_{1,t}, \dots, r_{m,t}$ of the polynomial $D_{m,t}(\lambda)$ converge, as $t \rightarrow \infty$, to the eigenvalues $\lambda_1, \dots, \lambda_m$ of the operator A . Moreover,

$$|r_{i,t} - \lambda_i| \leq C \left(\frac{\gamma}{\lambda_i} \right)^{2t} \quad (i = 1, \dots, m),$$

where C does not depend on t .

Received
5 VII 1956

REFERENCES

1. N. M. Krylov, N. N. Bogolyubov, *Izv. AN SSSR, OMEN*, No. 5, 471 (1929).
2. N. I. Akhiezer, M. G. Krein, *On Certain Questions in the Theory of Moments*, Kharkov, 1938.
3. J. Jackson, *Fourier Series and Orthogonal Polynomials*, IL, 1948, p. 174.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.