THE ALGEBRA OF MATRICES IN THE THEORY OF PARTICLES WITH SPIN \(3/2\)*

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Abstract

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THE ALGEBRA OF MATRICES IN THE THEORY OF PARTICLES WITH SPIN 3/2*

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1. The convenience of the algebraic method, when the matrices entering into the wave equation can be operated with as abstract numbers, knowing only the corresponding permutation relations, has been fully appreciated in the theory of particles with spin 1/2 (Dirac algebra) ([1]) and particles with spins 0 and 1 (Duffin–Kemmer algebra) ([2]). In the present paper we establish the basic relations of the algebra** of the α^k matrices entering into the wave equation for a particle with a single spin 3/2 and positive-definite total charge,

$$\left\{\alpha^k \frac{\partial}{\partial x_k} + i\chi\right\} \Psi(x_0, x_1, x_2, x_3) = 0, \tag{1}$$

where $\Psi(x_0, x_1, x_2, x_3) \equiv \Psi(t, x, y, z)$ is the 16-component wave function of a particle with spin 3/2; α^k are 16-dimensional square matrices, the explicit form of which in various representations is given in ([3–5]); χ is the mass of the particle***.

2. For particles with arbitrary spin, in ([6,7]) a general method was developed for deriving the permutation relations obtained from the form of the minimal equation satisfied by the corresponding matrices. The minimal equation for the matrices of a particle with a single definite spin s has, as is known ([6,8]), the form

$$(L^n)(L^2 - 1) = 0. \tag{2}$$

For particles with spins $1/2, 1, 3/2$, or 2, $n = 2s - 1$****.

However, the permutation relations following from (2) turn out to be too general. Thus, for particles with spin 1 (as well as for particles with spin 0) the following permutation relations are obtained:

$$\sum_p \beta^k(\beta^l\beta^m - g^{lm}) = 0, \tag{3}$$

where g^{lm} is the metric tensor of the special theory of relativity; $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$; $g^{lm} = 0$ $(l \neq m)$; $\sum_p$ denotes summation over all possible permutations of the indices k, l, m. The permutation relations (3) split into two types of permutation relations with symmetrization over two indices. One of these types, namely the Duffin–Kemmer permutation relations ([2]), is satisfied and determines the whole algebra of matrices of the theory of particles with spin 0 or 1.

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** Here and below, Latin indices run over the values $0, 1, 2, 3$, and Greek indices over $1, 2, 3$. As usual, summation is implied over identical indices occurring twice in one expression.

*** A system of units is chosen in the article in which $\hbar$, Planck' s constant, and c, the speed of light, are equal to one.

**** For particles with spin 2, the matrix L^0, constructed by the method of Gel' fand and Yaglom ([3]), satisfies the minimal equation $(L^0)^5 = (L^0)^3$.

3. For a particle with spin 3/2, from (2) the commutation relations follow*

$$\sum_p \alpha^i \alpha^k \left(\alpha^l \alpha^m - g^{lm}\right) = 0. \tag{4}$$

Here, in $\sum_p$, the permutation must be carried out over all four indices.

None of the three types of commutation relations with partial symmetrization over two or three indices, nor their linear combination, is satisfied by the matrices α^k. On the other hand, the general commutation relation (4) is satisfied not only for the matrices α^k of the theory of particles with spin 3/2, but also for the Dirac matrices γ^k, the Duffin–Kemmer matrices β^k, the matrices of the theory of a particle with mass χ and two spin states 3/2 and 1/2, given in work ([9]), and many others. Therefore, in order to single out precisely the algebra of matrices describing a particle with spin 3/2, it is necessary to find certain additional relations which, along with (4), would be satisfied for the matrices α^k of interest to us. The desired relations must meet the following requirements: a) they must not be satisfied by Hermitian matrices, since in the theory of particles with spin 3/2 and positive charge it is essential that the matrices α^k be non-Hermitian ([3]); b) from them there must follow additional conditions imposed on the wave function.

Such relations have the form**

$$g_{mn}\{\alpha^m \alpha_j \alpha^n + \alpha^n \alpha_j \alpha^m + \alpha_j(\alpha^m \alpha^n + \alpha^n \alpha^m) -$$

$$-(\alpha^m\alpha^n+\alpha^n\alpha^m)\alpha_j\}\{\alpha^k\alpha^l+\alpha^l\alpha^k\}=0. \tag{5}$$

The relations (5) were verified by us for the matrices α^k in the Petiau parametric representation ([5]). Consequently, they also hold in other representations, since, as shown in work ([11]), the matrices in the Petiau representation are connected by linear admissible transformations with other known representations of the matrices α^k ([3,4]).

For the complete construction of the matrix algebra of the theory of particles with spin 3/2, it is essential that the infinitesimal transformations I^{kl}, determining the transformation of the wave function Ψ,

$$\Psi\to\Psi'=\Psi+\frac{1}{2}\varepsilon_{kl}I^{kl}\Psi \tag{6}$$

under an infinitesimal Lorentz transformation

$$x^i\to x'^i=x^i+g^{ik}\varepsilon_{kl}x^l \qquad (\varepsilon_{kl}=-\varepsilon_{lk}) \tag{7}$$

can be represented through antisymmetric combinations of the matrices α^k

$$I^{kl}=\frac{3}{4}\{(\alpha^k\alpha^l-\alpha^l\alpha^k)+(\alpha^k\alpha^i\alpha_i\alpha^l-\alpha^l\alpha^i\alpha_i\alpha^k)-$$

$$-\frac{3}{4}(\alpha^i\alpha^k\alpha_i\alpha^l-\alpha^l\alpha^i\alpha_i\alpha^k)-\frac{5}{4}(\alpha^k\alpha^l-\alpha^l\alpha^k)\alpha^i\alpha_i+$$

$$+\frac{7}{4}(\alpha^k\alpha^i\alpha^l\alpha_i-\alpha^l\alpha^i\alpha^k\alpha_i)\}. \tag{8}$$

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* We note that the 20-dimensional matrices found in work ([10]), which can also be assigned to a particle with spin 3/2, satisfy only the minimal equation (2) with $n=3$ and, consequently, do not satisfy the commutation relation (4).

** It should be noted that the 16-dimensional matrices of the theory of a particle with spin 3/2 are, for relation (5), matrices of least dimension, since in work ([3]) it is shown that such matrices are unique and cannot be of lower dimension. The question of representations of relation (5) of higher dimension has not been investigated.

As is to be expected, the matrices α^k satisfy the relation ensuring the Lorentz invariance of the theory [3]

$$\alpha^kI^{lm}-I^{lm}\alpha^k=g^{kl}\alpha^m-g^{km}\alpha^l. \tag{9}$$

The transformation T corresponding to reflection

$$x^0 \to x^0, \qquad x^\nu \to x^\nu, \tag{10}$$

has the form

$$T = {}^3/_4\{\alpha^i\alpha_i\alpha^0 + \alpha^0\alpha^i\alpha_i - {}^8/_3\alpha^0 - 4(\alpha^0)^3\}, \qquad T^2 = 1. \tag{11}$$

4. In order to construct an invariant real Lagrange function, it is necessary to determine the basic invariant Hermitian bilinear form $\Psi^+\Psi = \Psi^*\eta\Psi$ *. Harish-Chandra [9] showed that the metric Gram matrix η, satisfying the conditions [9]

$$(\alpha^k)^*\eta - \eta\alpha^k = 0, \qquad I^{0\mu}\eta - \eta I^{0\mu} = 0, \tag{12}$$

$$T\eta - \eta T = 0, \qquad I^{\nu\mu}\eta - \eta I^{\nu\mu} = 0$$

must be sought in the form

$$\eta = \Lambda T, \tag{13}$$

where Λ is a certain scalar matrix, i.e. $I^{ik}\Lambda - \Lambda I^{ik} = 0, \quad T\Lambda - \Lambda T = 0$. In our case Λ is a linear combination of two scalar matrices present in the theory, I and $\alpha^i\alpha_i$ **:

$$\Lambda = (7 - 4\sqrt{3})\left(I - {}^3/_2(2 + \sqrt{3})\,\alpha^i\alpha_i\right). \tag{13'}$$

In doing this it must be taken into account that the matrices α^k are not Hermitian:

$$(\alpha^k)^* = \varepsilon^k\left\{\alpha^k - \frac{3\sqrt{3}}{2}\left(\alpha^i\alpha_i\alpha^k - \alpha^k\alpha^i\alpha_i - \frac{2}{\sqrt{3}}\alpha^i\alpha^k\alpha_i\right)\right\}, \tag{14}$$

where ε^k is a function giving the sign: $\varepsilon^0 = 1,\ \varepsilon^1 = \varepsilon^2 = \varepsilon^3 = -1$.

It should be borne in mind that the expressions for η and $(\alpha^k)^*$ in terms of α^k are valid only in the Petras representation, since upon passage to other representations related to the Petras representation by admissible but nonunitary transformations [11], η and $(\alpha^k)^*$ transform differently from α^k.

5. Having constructed the Lagrange function and introduced into it, in the usual way, the interaction with the electromagnetic field, after varying it we obtain the equations for a particle of spin 3/2 in an external electromagnetic field

$$(\alpha^l \pi_l + i\chi)\Psi = 0, \tag{15}$$

where $\pi_l = \dfrac{\partial}{\partial x_l} - ie\varphi_l$; φ_l is the four-dimensional potential of the electromagnetic field; e is the charge.

Using the relations (5), one can obtain two types of supplementary conditions imposed on the wave function. To derive the first type of supplementary conditions, which relate the components of the wave function

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* Here and below the asterisk denotes Hermitian conjugation.

** The numerical coefficient in formula (13′) is chosen from the condition that the metric matrix η has 12 eigenvalues, belonging to the spin-3/2 representation, equal to ± 1.

among themselves, we obtain from equation (15) the following second-order equation:

$$\left\{ \frac{1}{2} \left(\alpha^k \alpha^l + \alpha^l \alpha^k \right) \pi_k \pi_l - \frac{i}{4} e \left(\alpha^k \alpha^l - \alpha^l \alpha^k \right) F_{kl} + \chi^2 \right\} \Psi = 0. \tag{16}$$

Here we have used the commutation relations for π_k

$$\pi_k \pi_l - \pi_l \pi_k = -ieF_{kl},$$

where

$$F_{kl} = \frac{\partial \varphi_l}{\partial x_k} - \frac{\partial \varphi_k}{\partial x_l}$$

is the electromagnetic-field tensor.

Multiplying equation (16) by the matrix vector

$$A^j = \alpha^i \alpha^j \alpha_i + \alpha^j \alpha^i \alpha_i - \alpha^i \alpha_i \alpha^j, \tag{17}$$

by virtue of relations (5) we obtain additional conditions in the form

$$\left\{ \frac{1}{4i} \left(\frac{e}{\chi^2} \right) Q^{jkl} F_{kl} + A^j \right\} \Psi = 0, \tag{18}$$

where the antisymmetric tensor of rank three is

$$Q^{jkl} = A^j\left(\alpha^k\alpha^l - \alpha^l\alpha^k\right) = \left(\alpha^i\alpha^j\alpha_i + \alpha^j\alpha^i\alpha_i - \alpha^i\alpha_i\alpha^j\right)\left(\alpha^k\alpha^l - \alpha^l\alpha^k\right). \quad (19)$$

To derive additional conditions with derivatives of the wave function, equation (15) must be multiplied by the matrix tensor of rank two

$$R^{jk} = A^j\alpha^k = \left(\alpha^i\alpha^j\alpha_i + \alpha^j\alpha^i\alpha_i - \alpha^i\alpha_i\alpha^j\right)\alpha^k. \quad (20)$$

By virtue of (5) we obtain relativistically covariant conditions for Ψ

$$\left\{\frac{1}{2i}\left(\frac{1}{\chi}\right)Q^{jkl}\pi_l + R^{jk}\right\}\Psi = 0. \quad (21)$$

Since $Q^{jkl} = -Q^{jlk}$, the term $Q^{jkk} = 0$ (no summation over k) and, in equations (21), the term with π_k is absent. For $k = 0$ the term with the time derivative is absent, and equations (21) acquire the character of additional conditions connecting the spatial derivatives of the wave function with the wave function itself.

For a free particle $\varphi_l = 0$ and $F_{kl} = 0$, and the additional conditions (18) and (21) take the form

$$A^j\Psi = 0, \qquad \left\{\frac{1}{2i}\left(\frac{1}{\chi}\right)Q^{jkl}\frac{\partial}{\partial x_l} + R^{jk}\right\}\Psi = 0. \quad (22)$$

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Note: Figure translations are in progress. See original paper for figures.