# MATHEMATICS

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## Abstract

## Full Text

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# STURM-TYPE THEOREMS FOR SELF-ADJOINT SYSTEMS OF DIFFERENTIAL EQUATIONS OF HIGHER ORDER

*(Presented by Academician A. N. Kolmogorov on 23 I 1957)*

Consider the system

$$\sum_{i=0}^{n}(-1)^i \frac{d^i}{dt^i}\left(\theta_{n-i}\frac{d^i y}{dt^i}\right)=0, \qquad \theta_0>0, \tag{1}$$

where $\theta_j$ are continuous symmetric matrices of order $p$; $y$ is a $p$-dimensional vector.

The points $a$ and $b$ are called **joined** if there exists a solution $y(t)\not\equiv 0$ of system (1) such that

$$y(a)=y'(a)=\dots=y^{(n-1)}(a)=0, \qquad y(b)=y'(b)=\dots=y^{(n-1)}(b)=0. \tag{2}$$

The points $a$ and $b$ are called **conjugate** if they are joined and, moreover, in the interval $(a,b)$ there is no point joined to $a$.

Let $\tau_0$ be an arbitrary point, $\tau_0<\tau_1$, and let $\tau_1$ be conjugate to $\tau_0$; in general, let $\tau_k<\tau_{k+1}$, and let $\tau_{k+1}$ be conjugate to $\tau_k$. Analogously, let $\tau_{-1}<\tau_0$, and let $\tau_{-1}$ be conjugate to $\tau_0$, and, in general, let $\tau_{-k-1}<\tau_{-k}$, and let $\tau_{-k-1}$ be conjugate to $\tau_{-k}$. The system of points thus obtained is called a **system of conjugate points**.

The following general theorems are proved:

**Theorem 1.** *Two systems of conjugate points that do not coincide separate each other, i.e. between two consecutive points of one system there exists one and only one point of the other.*

**Theorem 2.** *Let the system*

$$\sum_{i=0}^{n}(-1)^i \frac{d^i}{dt^i}\left(\overline{\theta}_{n-i}\frac{d^i y}{dt^i}\right)=0$$

*be given. If $\overline{\theta}_{n-i} \geq \theta_{n-i}$ and $\overline{\tau}_0 = \tau_0$, then $\overline{\tau}_k \geq \tau_k$, $\overline{\tau}_{-k} \leq \tau_{-k}$; moreover, if for some $l$, $\overline{\theta}_{n-l} > \theta_{n-l}$, then the inequalities become strict.*

**Theorem 3.** *Let the system*

$$\sum_{i=0}^{n}(-1)^i \frac{d^i}{dt^i}\left(\theta_{n-i}\frac{d^i y}{dt^i}\right) - \lambda \sum_{i=0}^{n-1}(-1)^i \frac{d^i}{dt^i}\left(p_i \frac{d^i y}{dt^i}\right) = 0, \tag{3}$$

*be given, where $p_j \geq 0$, and for at least one $j$ we have $p_j > 0$.*

a) *There exists a sequence of numbers $\lambda_1, \lambda_2, \dots, \lambda_k$, $\lim_{n\to\infty} \lambda_n = \infty$, such that for $\lambda = \lambda_k$ there is a system of $k+1$ conjugate points, the first of which coincides with $a$, and the last with $b$.*

b) *There exists a sequence of eigenvalues of problem (3), (2), $\lambda'_1, \lambda'_2, \dots$, such that for $\lambda = \lambda'_k$, $b$ is the $k$-th point joined to the point $a$.*

These general theorems are proved in two ways. The first way uses the method of V. B. Lidskii ([1]) and the arguments of Bliss and Schoenberg ([2]). The second way uses the minimum of the functional

$$\int_a^b \left[(\theta_0 y^{(n)}, y^{(n)}) + (\theta_1 y^{(n-1)}, y^{(n-1)}) + \dots + (\theta_n y, y)\right] dt \tag{4}$$

in the class of functions satisfying the conditions (2), and

$$\int_a^b \left[(p_{n-1} y^{(n-1)}, y^{(n-1)}) + \dots + (p_0 y, y)\right] dt = 1.$$

The existence of this minimum is proved by Ritz' s method according to the same scheme as in ([3]), and after this the arguments almost literally repeat the known proofs for the case of a second-order equation ([4]).

We shall say that the case of **nonoscillation** occurs if there exists $t_0$ such that on the half-axis $t > t_0$ there is no pair of conjugate points. Using the comparison theorem, the equivalent theorem on the matrix Riccati equation ([5]), and also the functional (4), one can obtain nonoscillation criteria generalizing the known criteria for an equation of second order. As an example we give several such criteria, taking, for simplicity, $n = 2$.

For nonoscillation it is sufficient that one of the following conditions be fulfilled:

1) $\theta_0 \geqslant E_p$, $\theta_1 \geqslant -\frac{5}{2t^2}E_p$, $\theta_2 \geqslant \frac{81}{16t^4}E_p$, where $E_p$ is the identity matrix of order $p$.

2)
$$\begin{pmatrix} -\mu_2 & \Pi_2 \\ \Pi_2 & -\mu_1 + \Pi_1 \theta_0^{-1} \Pi_1 \end{pmatrix} \leqslant 0,$$

where $\theta_1 = \mu_1 + \Pi_1',\ \theta_2 = \mu_2 + \Pi_2'$.

3) $\theta_2 \geqslant 0$, there exists $\int_t^\infty \theta_1\, dt$, and

$$-\left[4\int_{t_0}^{t} \beta(t)\, dt\right]^{-1} \leqslant \int_t^\infty \theta_1\, dt \leqslant 3\left[4\int_{t_0}^{t} \beta(t)\, dt\right]^{-1} E_p,$$

where $\beta(t)$ is the largest eigenvalue of the matrix $\theta_0^{-1}(t)$.

With the aid of the functional (4) one can also obtain necessary conditions for nonoscillation.

We indicate such a condition for $n = 2$. If the case of nonoscillation occurs and $\theta_0 \leqslant E_p$, then for every $\alpha > 0$ we have either $\theta_1 > \dfrac{5+2\alpha}{2t^2} E_p$, or $\theta_2 > \dfrac{9(9+\alpha)}{16t^4} E_p$.

The results obtained generalize the results of Cimmino ([6]) and of Bliss and Schoenberg ([2]).

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*