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THEORY OF ELASTICITY

1957

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Abstract

Full Text

THEORY OF ELASTICITY

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ON THE PROBLEM OF THE ELASTIC EQUILIBRIUM OF BODIES OF REVOLUTION

(Presented by Academician V. I. Smirnov, 23 XI 1956)

A certain body of revolution is referred to the cylindrical coordinate system $r\varphi z$. The z-axis is made to coincide with the axis of revolution. We shall find a solution of the equations of the theory of elasticity when the radial (u), tangential (v), and axial (w) displacements are prescribed in the form

$$u = u_m(r,z)\cos m\varphi, \qquad v = v_m(r,z)\sin m\varphi, \qquad w = w_m(r,z)\cos m\varphi. \quad (1)$$

Corresponding to this prescription of the displacements, the volume strain ϑ and the components of rotation ω_r, ω_φ, and ω_z have the form

$$\vartheta = \vartheta_m(r,z)\cos m\varphi, \qquad \omega_r = \omega_{rm}(r,z)\sin m\varphi,$$

$$\omega_\varphi = \omega_{\varphi m}\cos m\varphi, \qquad \omega_z = \omega_{zm}(r,z)\sin m\varphi, \quad (2)$$

where ϑ_m, ω_{rm}, $\omega_{\varphi m}$, and ω_{zm} are functions of the two variables r and z, related to the functions u_m, v_m, and w_m by the equalities

$$\vartheta_m = \frac{\partial u_m}{\partial r} + \frac{u_m + m v_m}{r} + \frac{\partial w_m}{\partial z}, \qquad 2\omega_{rm} = -\frac{m}{r} w_m - \frac{\partial v_m}{\partial z},$$

$$2\omega_{\varphi m} = \frac{\partial u_m}{\partial z} - \frac{\partial w_m}{\partial r}, \qquad 2\omega_{zm} = \frac{\partial v_m}{\partial r} + \frac{m u_m + v_m}{r}. \quad (3)$$

Substitution of (2) into the equations of statics in displacements leads to the following system of equations:

$$(\lambda + 2\mu)\frac{\partial \vartheta_m}{\partial r} - 2\mu\left(\frac{m}{r}\omega_{zm} - \frac{\partial \omega_{\varphi m}}{\partial z}\right) = 0,$$

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$$(\lambda+2\mu)\frac{m}{r}\vartheta_m - 2\mu\left(\frac{\partial\omega_{rm}}{\partial z} - \frac{\partial\omega_{zm}}{\partial r}\right) = 0, \tag{4}$$

$$(\lambda+2\mu)\frac{\partial\vartheta_m}{\partial z} - 2\mu\frac{1}{r}\left(\frac{\partial r\omega_{\varphi m}}{\partial r} - m\omega_{rm}\right) = 0$$

(λ and μ are Lamé constants).

These equations are satisfied if one sets

$$\vartheta_m = \frac{1-2\nu}{\mu}\frac{1}{r^{m+1}}\frac{\partial^2}{\partial r\,\partial z}(f_m - \psi_m),$$

$$\omega_{rm} = \frac{1}{2\mu}\frac{m}{r^{m+1}}\left[\left(\frac{\partial^2}{\partial z^2} - \frac{m}{r}\frac{\partial}{\partial r}\right)f_m - \frac{2(1-\nu)}{r}\frac{\partial}{\partial r}(f_m - \psi_m)\right], \tag{5}$$

$$\omega_{\varphi m} = \frac{1}{2\mu}\frac{1}{r^{m+1}}\left[2(1-\nu)\left(\frac{\partial^2}{\partial z^2} - \frac{m}{r}\frac{\partial}{\partial r}\right)(f_m - \psi_m) - \frac{m^2}{r}\frac{\partial f_m}{\partial r}\right],$$

$$\omega_{zm} = -\frac{1}{2\mu}\frac{m}{r^{m+1}}\frac{\partial^2 f_m}{\partial r\,\partial z}.$$

(ν is Poisson' s ratio) and to determine the functions ψ_m and f_m by second-order differential equations

$$\nabla^2_{2m+1}\psi_m = 0, \qquad \nabla^2_{2m+1}f_m = 0, \tag{6}$$

where

$$\nabla^2_{2m+1} = \frac{\partial^2}{\partial r^2} - \frac{2m+1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

The introduction of a new auxiliary function Ψ_m by the conditions

$$\frac{\partial v_m}{\partial r} - \frac{mu_m + v_m}{r} = -\frac{m}{\mu r^{m+2}}\left[\frac{\partial^2\Psi_m}{\partial z^2} - \frac{m-1}{r}\frac{\partial\Psi_m}{\partial r} - 2\frac{\partial\psi_m}{\partial z} + (m+1)\frac{\partial f_m}{\partial z}\right],$$

$$\frac{\partial v_m}{\partial z} - \frac{m}{r}w_m = \frac{m}{\mu r^{m+2}}\left[\frac{\partial^2\Psi_m}{\partial r\partial z} + m\frac{\partial f_m}{\partial r} + 2(1-\nu)\frac{\partial}{\partial r}(f_m - \psi_m)\right] \tag{7}$$

leads to the following values of the displacements u, v, and w:

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$$u = \frac{1}{2\mu}\frac{1}{r^{m+1}}\left(\frac{\partial^2\Psi_m}{\partial r^2} - \frac{m+1}{r}\frac{\partial\Psi_m}{\partial r} - m\frac{\partial f_m}{\partial z} - r\frac{\partial^2 f_m}{\partial r\partial z}\right)\cos m\varphi,$$

$$v = -\frac{m}{2\mu}\frac{1}{r^{m+1}}\left(\frac{1}{r}\frac{\partial\Psi_m}{\partial r} + \frac{\partial f_m}{\partial z}\right)\sin m\varphi, \tag{8}$$

$$w = \frac{1}{2\mu}\frac{1}{r^{m+1}}\left[\frac{\partial^2\Psi_m}{\partial r\partial z} + 2m\frac{\partial f_m}{\partial r} + 4(1-\nu)\frac{\partial}{\partial r}(f_m - \psi_m) - r\frac{\partial^2 f_m}{\partial z^2}\right]\cos m\varphi.$$

For the integration of equations (3) and (7) to be possible, the function Ψ_m must be connected with the function ψ_m by the equality

$$\Psi_m = \chi_m + z\psi_m,$$

where $\chi_m(r,z)$ is a function satisfying the equation

$$\nabla^2_{2m+1}\chi_m = 0.$$

After substituting (8) into Hooke' s equations for the stresses, we obtain

$$\sigma_r = \left(\frac{\partial^2}{\partial r^2}\left(\frac{1}{r^{m+1}}\frac{\partial\Psi_m}{\partial r} - \frac{1}{r^m}\frac{\partial f_m}{\partial z}\right) + \frac{2}{r^{m+1}}\frac{\partial}{\partial z}\left\{\frac{m(m+1)}{r}f_m - \frac{\partial}{\partial r}[(m-\nu)f_m + \nu\psi_m]\right\}\right)\cos m\varphi,$$

$$\sigma_\varphi = \left\{\frac{1}{r}\left(\frac{\partial}{\partial r} - \frac{m^2}{r}\right)\left(\frac{1}{r^{m+1}}\frac{\partial\Psi_m}{\partial r} - \frac{1}{r^m}\frac{\partial f_m}{\partial z}\right) - \frac{2}{r^{m+1}}\frac{\partial}{\partial z}\left[\frac{m(m+1)}{r}f_m - \nu\frac{\partial}{\partial r}(f_m - \psi_m)\right]\right\}\cos m\varphi,$$

$$\sigma_z = \frac{1}{r^{m+1}}\frac{\partial^2}{\partial r\partial z}\left\{\frac{\partial\Psi_m}{\partial z} - 2(2-\nu)\psi_m + 2(1-\nu)f_m + r\frac{\partial f_m}{\partial r}\right\}\cos m\varphi, \tag{9}$$

$$\tau_{r\varphi} = -m\left[\frac{\partial}{\partial r}\left(\frac{1}{r^{m+2}}\frac{\partial\Psi_m}{\partial r}\right) - \frac{m+1}{r^{m+2}}\frac{\partial f_m}{\partial z}\right]\sin m\varphi,$$

$$\tau_{\varphi z} = -\frac{m}{r^{m+2}}\frac{\partial}{\partial r}\left[\frac{\partial\Psi_m}{\partial r} + mf_m + 2(1-\nu)(f_m - \psi_m)\right]\sin m\varphi,$$

$$\tau_{zr} = \left(\frac{\partial^2}{\partial r\partial z}\left(\frac{1}{r^{m+1}}\frac{\partial\Psi_m}{\partial r} - \frac{1}{r^m}\frac{\partial f_m}{\partial z}\right) - \frac{m^2}{r^{m+2}}\frac{\partial f_m}{\partial r} + 2\frac{\partial}{\partial r}\left\{\frac{1}{r^{m+1}}\frac{\partial}{\partial r}[(m+1-\nu)f_m - (1-\nu)\psi_m]\right\}\right)\cos m\varphi.$$

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The obtained form of the solution is convenient in considering problems on the loading of bodies of revolution by forces represented by Fourier-series expansions in the variable φ:

$$p_r = \sum_{m=0}^{\infty} p_{rm} \cos m\varphi, \qquad p_\varphi = p_{\varphi 0} + \sum_{m=1}^{\infty} p_{\varphi m} \sin m\varphi, \qquad p_z = \sum_{m=0}^{\infty} p_{zm} \cos m\varphi. \tag{10}$$

The corresponding values of the displacements and stresses also have the form of series*. Fulfillment of the conditions of static equivalence of the stresses to the external forces in each cross section of the body of revolution is achieved while retaining only the first two terms of these series. The value $m = 0$ corresponds to torsion and axisymmetric loading, and $m = 1$ to bending of bodies of revolution under a definite distribution of the external forces over the surface.

Interchanging cosine and sine in (8) and (9) and putting $m = 0$, after natural substitutions simplifying the calculations, one easily relates the functions introduced here to Michell' s stress function [(5)] for torsion of shafts of variable diameter. Equating m to zero (without interchanging cosine and sine) leads to the solution of the axisymmetric problem in the form indicated in [(6)]. From formulas (8) and (9), for $m = 1$, follow expressions for the displacements and stresses in the bending of circular bars of variable cross section by loads distributed according to sine and cosine laws along the contour of the cross section [(7)].

In each of the cases indicated above, the values of the stress functions are determined by the solutions of the differential equations (6) for $m = 0$ and $m = 1$. By means of the contour conditions these functions are simply related to the magnitudes of the twisting moment (Michell' s function), the normal force (the function of [(6)]), the shearing force, and the bending moment (the functions of [(7)]).

The solutions for the values $m = 2, 3, 4, ...$ identically satisfy the conditions of static equivalence. Appending to them the solutions of the problems of torsion, axisymmetric loading, and bending (in the sense indicated above) makes it possible to satisfy exactly the contour conditions of the problem for any distribution of load on the surface of the body of revolution**.

Solving equation (6) in the form of a product of two functions $f_m(r, z) = R(r) \cdot Z(z)$ leads to the following values:

$$f_{mk} = [A_k e^{kz} + B_k e^{-kz}] \, [C_k J_{m+1}(kr) + D_k Y_{m+1}(kr)] \, r^{m+1}; \tag{11}$$

$$f_{mk} = [A_k \sin kz + B_k \cos kz] \, [C_k I_{m+1}(kr) + D_k K_{m+1}(kr)] \, r^{m+1}. \tag{12}$$

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The transformation of (6) to curvilinear orthogonal coordinates ξ and η, taken in the plane of the axial section of the body of revolution, gives

$$\frac{\partial}{\partial \xi}\left(\frac{1}{r^{2m+1}}\frac{\partial f_m}{\partial \xi}\right)+\frac{\partial}{\partial \eta}\left(\frac{1}{r^{2m+1}}\frac{\partial f_m}{\partial \eta}\right)=0. \tag{13}$$

In the case where ξ and η are polar coordinates in the plane rOz, the solution of equation (13) in the form $f_m(\xi,\eta)=f_1(\xi)f_2(\eta)$ has the form

$$f_{mk}(\xi,\eta)=[A_k e^{(m+k+1)\xi}+B_k e^{(m-k)\xi}][C_k R_k^{(m)}(y)+D_k S_k^{(m)}(y)] \tag{14}$$

($y=\cos\eta$), where $R_k^{(m)}$ and $S_k^{(m)}$ are determined by the equalities

$$R_k^{(m)}=1+\frac{1}{2!}(m-k)(m+k+1)y^2+$$

$$+\frac{1}{4!}(m-k)(m-k-2)(m+k+1)(m+k-1)y^4+\cdots$$

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* A solution of similar problems for solid and hollow cylinders is contained in works [1–4].

** The practical realization of what is indicated is connected with computational difficulties analogous to the difficulties in solving the axisymmetric problem for a cylinder.

$$\ldots+\frac{1}{(2s)!}(m-k)\ldots[m-k-2(s-1)](m+k+1)\ldots$$

$$\ldots[m+k-(2s-3)]y^{2s}+\ldots,$$

$$S_k^{(m)}=y+\frac{1}{3!}(m-k-1)(m+k)y^3+$$

$$+\frac{1}{5!}(m-k-1)(m-k-3)(m+k)(m+k-2)y^5+\ldots$$

$$\ldots+\frac{1}{(2s+1)!}(m-k-1)\ldots[m-k-(2s-1)](m+k)\ldots$$

$$\ldots[m+k-2(s-1)]y^{2s+1}+\ldots.$$

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If $k > m+1$, the functions $R_k^{(m)}$ and $S_k^{(m)}$ can be related to derivatives of the spherical Legendre functions $P_k^{(m+1)}(y)$ and $Q_k^{(m+1)}(y)$, after which the solution (14) takes the form:

$$f_{mk}(\xi,\eta) = \left[A_k e^{(m+k+1)\xi} + B_k e^{(m-k)\xi}\right] \times$$

$$\times \left[C_k P_k^{(m+1)}(y) + D_k Q_k^{(m+1)}(y)\right] (\sin\eta)^{2(m+1)}. \tag{15}$$

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Received
16 XI 1956

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Note: Figure translations are in progress. See original paper for figures.

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