HYDROMECHANICS

S. A. REGIRER

1957

SovietRxiv

Abstract

Full Text

HYDROMECHANICS

S. A. REGIRER

ON THE UNIQUENESS OF THE SOLUTION OF APPROXIMATE BOUNDARY-VALUE PROBLEMS IN THE DYNAMICS OF AN INCOMPRESSIBLE FLUID WITH VARIABLE VISCOSITY

(Presented by Academician S. L. Sobolev, 31 V 1957)

1. The equations of the dynamics of an incompressible fluid with variable viscosity have the form ([1])

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{F} + \nu\Delta\mathbf{v} + 2(\vec{\alpha}\nabla)\mathbf{v} + \vec{\alpha}\times\vec{\Omega}; \tag{1}$$

$$\operatorname{div}\mathbf{v} = 0; \tag{2}$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = a\Delta T + \frac{\nu}{Ic}E, \tag{3}$$

where $E = [\Omega^2 + 2\operatorname{div}(\mathbf{v}\nabla)\mathbf{v}]$, $\vec{\alpha} = \nabla\nu$, $\vec{\Omega} = \operatorname{rot}\mathbf{v}$ (the remaining notation is standard). In solving problems of flow and heat exchange in a stream when $\nu = \nu(T)$, two approximate methods are encountered: a) for a prescribed temperature distribution, from equations (1), (2) one finds $\mathbf{v}$ and p; b) for a prescribed velocity distribution, from (3) one determines T. Successive alternating application of both methods constitutes a certain system of approximations; moreover, the question of convergence of the process and of obtaining in this way a solution of the system (1)–(3) requires special consideration. The determination of these successive approximations is reduced to the solution of a number of similar boundary-value problems, the uniqueness of which is proved in the present note. For this purpose we shall use the method of D. E. Dolidsze ([2,3]).

2. Let, in equations (1), (2), the function $\nu(x, y, z, t)$, continuous and bounded together with its first and second derivatives with respect to the coordinates in $D+B$, be determined uniquely, where B is a piecewise smooth surface bounding the domain D.

For the sought function $\mathbf{v}(x, y, z, t)$ the following conditions are prescribed: at $t = 0$, $\mathbf{v}_0(x, y, z)$ is prescribed in D; for $t > 0$, on the boundary B the velocity $\mathbf{v}$ assumes prescribed continuous values. If D is an exterior domain, then $\mathbf{v}$ also satisfies conditions at infinity.

We shall prove that equations (1), (2) under these conditions admit a unique solution for $\mathbf{v}$, regular in D and having in $D + B$ continuous first derivatives of the components of $\mathbf{v}$ with respect to the coordinates.

Assume the contrary—that there exist two solutions $\mathbf{v}_1, p_1$ and $\mathbf{v}_2, p_2$. Denote

$$\mathbf{u} = \mathbf{v}_1 - \mathbf{v}_2, \qquad q = p_1 - p_2, \qquad \vec{\omega} = \vec{\Omega}_1 - \vec{\Omega}_2 = \operatorname{rot} \mathbf{u},$$

so that the boundary and initial values of $\mathbf{u}$ are zero. Substituting both solutions into (1), subtracting one of the resulting equations from the other, and···

multiplying the result scalarly by $\mathbf{u}$, after transformations we obtain

$$\frac{1}{2}\frac{\partial u^2}{\partial t} + \mathbf{u}\left[(\mathbf{v}_1\nabla)\mathbf{u}\right] + \mathbf{u}\left[(\mathbf{u}\nabla)\mathbf{v}_2\right] =$$

$$= -\frac{1}{\rho}\mathbf{u}\nabla q + \nu\, \mathbf{u}\Delta\mathbf{u} + 2\mathbf{u}\left[(\vec{\alpha}\nabla)\mathbf{u}\right] + \mathbf{u}\left(\vec{\alpha} \times \vec{\omega}\right). \tag{4}$$

With the aid of the relations (2)

$$\operatorname{div}(\varphi \mathbf{A}) = \varphi \operatorname{div} \mathbf{A} + \nabla\varphi \cdot \mathbf{A}, \tag{5}$$

$$\mathbf{A}[(\mathbf{B}\nabla)\mathbf{C}] + \mathbf{C}[(\mathbf{B}\nabla)\mathbf{A}] = \operatorname{div}[(\mathbf{AC})\mathbf{B}] - \mathbf{AC}\operatorname{div}\mathbf{B}, \tag{6}$$

and the condition $\operatorname{div}\mathbf{u} = 0$, which follows from (2), we reduce equation (4) to the form

$$\frac{\partial u^2}{\partial t} = \operatorname{div}\left[2u^2\vec{\alpha} - u^2\mathbf{v}_1 - \frac{2q\mathbf{u}}{\rho} + \nu\nabla u^2 + 2(\mathbf{u}\vec{\alpha})\mathbf{u}\right] -$$

$$-2\vec{\alpha}\nabla u^2 - 2\nu\sum_3 (\nabla u_i)^2 - 2\mathbf{u}\left[(\mathbf{u}\nabla)(\mathbf{v}_2 + \vec{\alpha})\right]. \tag{7}$$

Suppose that D is an interior domain and B is a closed surface. Integrating (7) over D and using Gauss’ s theorem, we obtain

$$\frac{d}{dt}\int_D u^2\, dD = -2\int_D (\vec{\alpha}\nabla u^2)\, dD - 2\int_D \nu\sum_3 (\nabla u_i)^2\, dD - 2\int_D \mathbf{u}\left[(\mathbf{u}\nabla)(\mathbf{v}_2 + \vec{\alpha})\right] dD.$$

The second term on the right is manifestly nonpositive; therefore

$$\frac{d}{dt}\int_D u^2\,dD \le -2\int_D (\vec{\alpha}\nabla u^2)\,dD - 2\int_D \mathbf{u}\,[(\mathbf{u}\nabla)(\mathbf{v}_2+\vec{\alpha})]\,dD.$$

Applying Green' s theorem, the mean-value theorem, and the estimate

$$\left|\int_D \mathbf{u}\,[(\mathbf{u}\nabla)(\mathbf{v}_2+\vec{\alpha})]\,dD\right| < M\sqrt{3}\int_D u^2\,dD, \tag{8}$$

where $M = \max\{|\nabla(v_{2i}+\alpha_i)|\}$, we arrive at the inequality

$$\frac{\partial}{\partial t}\int_D u^2\,dD \le N\int_D u^2\,dD, \tag{9}$$

where $N = 2[M\sqrt{3} + (\Delta\nu)_{\mathrm{av}}]$. It is not difficult to see that always, except in the case of oscillation of $\int_D u^2\,dD$ with infinite frequency, the left-hand side in (9) is nonnegative for $t \ge 0$. The fastest growth of $\int_D u^2\,dD$ in time is attained when

$$\frac{\partial}{\partial t}\int_D u^2\,dD = N\int_D u^2\,dD, \tag{10}$$

i.e., when $\int_D u^2\,dD = \mathrm{const.}\,e^{Nt}$.

Taking the initial condition into account, we obtain $\mathbf{u} \equiv 0$. From equation (1) it is then easy to find that $q = \mathrm{const}$, i.e., that the pressure is determined up to an additive constant.

3. Let us consider the second approximate problem, which consists in finding $T(x, y, z, t)$ from equation (3) under prescribed initial and boundary conditions for T of the same kind as the conditions for the velocity $\mathbf{v}$ in Sec. 2. With regard to $\mathbf{v}$, entering into (3), we assume that it satisfies equation (2) and the conditions of Sec. 2. We shall now prove that, under these restrictions and the additional requirement of boundedness of the derivative $d\nu/dT$ for all T, equation (3) has a unique regular solution in D.

Suppose the contrary: let there exist two such solutions T_1 and T_2, with $\vartheta = T_1 - T_2$. Substituting T_1 and T_2 into (3), subtracting one equation

from the other and multiplying by ϑ, we shall have

$$\frac{\partial\vartheta^2}{\partial t} + \mathbf{v}\nabla\vartheta^2 = 2a\vartheta\Delta\vartheta + 2\frac{\vartheta\mu}{Ic}E, \tag{11}$$

where $\mu = \nu(T_1) - \nu(T_2)$.

Using relation (5) and the theorems of Gauss and Green, and taking into account that $\vartheta = 0$ on B, we integrate (11) over D:

$$\frac{d}{dt}\int_D \vartheta^2\, dD = -a\int_D (\nabla\vartheta)^2\, dD + \frac{2}{Ic}\int_D \vartheta\mu E\, dD.$$

Hence follows the inequality

$$\frac{d}{dt}\int_D \vartheta^2\, dD \le \frac{2}{Ic}\left|\int_D \vartheta\mu E\, dD\right|. \tag{12}$$

Now we have

$$\int_D \vartheta\mu E\, dD = \int_D \vartheta^2 \frac{\mu}{\vartheta} E\, dD$$

or, by the mean-value theorems,

$$\int_D \vartheta\mu E\, dD = \left(\frac{d\nu}{dT}E\right)_{\text{av}} \int_D \vartheta^2\, dD. \tag{13}$$

Then (12) takes the form

$$\frac{d}{dt}\int_D \vartheta^2\, dD \le R\int_D \vartheta^2\, dD, \tag{14}$$

where

$$R = \frac{2}{Ic}\max\left|\frac{d\nu}{dT}E\right|.$$

Arguments analogous to those carried out in Sec. 2 lead to the conclusion that $\vartheta \equiv 0$.

4. The extension of the proofs of the theorems given in Secs. 2, 3 to the case when D is an exterior domain and B is an open surface is carried out in the same way as was done in the works [2,3]. In this case, at infinity the following decay conditions must be satisfied: for the theorem of Sec. 2,

$$\lim_{r\to\infty} r^{1+\alpha} v_i = 0, \qquad \lim_{r\to\infty} r^{1+\alpha}\frac{\partial v_i}{\partial x_j} = 0, \qquad \lim_{r\to\infty} r^{\alpha} q = 0;$$

for the theorem of Sec. 3,

$$\lim_{r\to\infty} r^{1+\alpha} T = 0, \qquad \lim_{r\to\infty} r^{1+\alpha} \frac{\partial T}{\partial x_i} = 0,$$

where $\alpha \geq \frac{1}{2}$.

It should be noted that the quantities N and R in formulas (10) and (14) are not necessarily constants. In the general case they may be unbounded functions of t, and then from (10), for example, we obtain

$$\int_D u^2\, dD = \text{const} \cdot \exp \int_0^t N(t)\, dt.$$

It is obvious that for the proof of the theorem the integrability of N and R with respect to t is necessary.

Vorkuta Scientific-Research Permafrost Station
of the V. A. Obruchev Institute of Permafrost Studies
Academy of Sciences of the USSR

Received
22 V 1957

CITED LITERATURE

1. L. M. Milne-Thompson, *Theoretical Hydrodynamics*, 3rd ed., London, 1955; W. F. Cope, *The Equations of Hydrodynamics in a Very General Form*, ARC Rep. and Mem., No. 1903 (1942).

2. D. E. Dolidze, DAN, 96, No. 3, 437 (1954).

3. D. E. Dolidze, Tr. Tbilissk. matem. inst., 21, 261 (1955).

Note: Figure translations are in progress. See original paper for figures.