ON A CONTINUAL ANALOGUE OF ONE CHRISTOFFEL FORMULA FROM THE THEORY OF ORTHOGONAL POLYNOMIALS

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Abstract

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MATHEMATICS

M. G. KREIN

ON A CONTINUAL ANALOGUE OF ONE CHRISTOFFEL FORMULA FROM THE THEORY OF ORTHOGONAL POLYNOMIALS

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1. Let $\varphi(r;\lambda)$ be a solution of the differential system

$$d^2\varphi/dr^2 - V(r)\varphi + \lambda\varphi = 0, \quad \varphi(0;\lambda) = 1, \quad \varphi'(0;\lambda) = h \quad (0 \le r < r_\infty), \; (S_{0,h})$$

where h is a certain real number; $V(r)$ $(0 \le r < r_\infty;\ r_\infty \le \infty)$ is a real continuous function; λ is a complex parameter.

We shall agree on the following notation. If the complex numbers α_j $(j = 1, 2, \dots, p)$ are all distinct from one another, then by the symbol

$$W_*(\varphi_{\alpha_1}, \varphi_{\alpha_2}, \dots, \varphi_{\alpha_p}) \tag{1}$$

we shall denote the function of r obtained from the formally constructed Wronskian determinant W for the functions $\varphi(r;\alpha_1), \dots, \varphi(r;\alpha_p)$ by replacing each even derivative $\varphi^{(2k)}(r;\alpha_j)$ by $(-\alpha_j)^k\varphi(r;\alpha_j)$ and each odd derivative $\varphi^{(2k+1)}(r;\alpha_j)$ by $(-\alpha_j)^k\varphi'(r;\alpha_j)$. It is easy to see that if the potential $V(r)$ is continuously differentiable at least $p-3$ times and, consequently, the function $\varphi(r;\lambda)$ at least $p-1$ times, then the expression (1) will coincide exactly with the ordinary Wronskian determinant W of the indicated system of functions.

Putting $\mathcal{P}(\lambda) = (\lambda-\alpha_1)(\lambda-\alpha_2)\cdots(\lambda-\alpha_p)$, we shall write:

$$V_{\mathcal{P}}(r) = V(r) - 2\frac{d^2}{dr^2}\ln W_*(\varphi_{\alpha_1}, \dots, \varphi_{\alpha_p}), \tag{2}$$

$$\varphi_{\mathcal{P}}(r;\lambda) = (-1)^p\mathcal{P}^{-1}(\lambda)\, W_*(\varphi_\lambda, \varphi_{\alpha_1}, \dots, \varphi_{\alpha_p})/W_*(\varphi_{\alpha_1}, \dots, \varphi_{\alpha_p}). \tag{3}$$

The case when some of the numbers α_j coincide with one another may be regarded as a limiting case relative to that in which all these numbers are distinct. Proceeding from this, it is not hard to see how the functions $V_{\mathcal{P}}(r)$ and

$\varphi_{\mathcal{P}}(r;\lambda)$ should naturally be defined in the case of an arbitrary polynomial $\mathcal{P}(\lambda)=\lambda^p+a_1\lambda^{p-1}+\cdots+a_p$.

It can be shown (cf. ([1])) that $V_{\mathcal{P}}(r)$, as $r\to 0$, has the behavior

$$V_{\mathcal{P}}(r)=p(p-1)/r^2+O(1).$$

As is known, for the system $(S_{0,h})$ there always exists at least one **spectral function** τ, i.e. a nondecreasing function $\tau(\lambda)=\tau(\lambda-0)$ $(-\infty<\lambda<\infty;\ \tau(-\infty)=0)$ such that for every function $f(r)\in L_2(0,r_\infty)$ that is equal to zero in some left neighborhood of the point r_∞,

$$\int_0^{r_\infty}|f(r)|^2\,dr=\int_{-\infty}^{\infty}\left|\int_0^{r_\infty}f(r)\varphi(r;\lambda)\,dr\right|^2 d\tau(\lambda). \tag{4}$$

Theorem 1. Let $\tau(\lambda)$ be some spectral function of the system $(S_{0,h})$, and let $\mathfrak{P}(\lambda)=\lambda^p+\cdots+a_p$ be a polynomial nonnegative on the spectrum of the function τ. Then the function

$$\tau_{\mathfrak{P}}(\lambda)=\int_{-\infty}^{\lambda}\mathfrak{P}(\mu)\,d\tau(\mu)$$

will be the spectral function of the differential system

$$\psi''-V_{\mathfrak{P}}(r)\psi+\lambda\psi=0,\qquad \lim_{r\downarrow 0}r^{-p}\psi(r;\lambda)=1/1\cdot 3\cdots(2p-1) \tag{5}$$

with continuous potential $V_{\mathfrak{P}}(r)$ $(0\le r<r_\infty)$, defined by formula (2). The solution $\psi(r;\lambda)$ of this system will be the function $\varphi_{\mathfrak{P}}(r;\lambda)$, defined by formula (3).

Thus it is asserted that equality (4) will not be violated if in it one replaces the function $\varphi(r;\lambda)$ by $\varphi_{\mathfrak{P}}(r;\lambda)$ and the function $\tau(\lambda)$ by $\tau_{\mathfrak{P}}(\lambda)$.

This assertion should be regarded as a continuous analogue of the well-known (suitably reformulated) Christoffel rule (see, for example, ([2]), p. 198) on the transformation undergone by orthogonal polynomials with respect to a certain weight when the latter is multiplied by a nonnegative polynomial.

2. Suppose, for example, that $r_\infty<\infty$ and $V(r)\in C(0,r_\infty)$ (or, more generally, $V(r)\in L_1(0,r_\infty)$). Then, adding to the system $(S_{0,h})$ some boundary condition

$$\varphi'(r_\infty;\lambda)+H\varphi(r_\infty;\lambda)=0$$

with real H ($\leq \infty$), we obtain a Sturm–Liouville boundary-value problem with some spectrum:

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots,$$

which at the same time will be the spectrum (set of points of increase) of some spectral (orthogonal) function τ of the system $(S_{0,h})$. If we put

$$\mathfrak{P}(\lambda) = (\lambda - \lambda_0)(\lambda - \lambda_1) \cdots (\lambda - \lambda_{p-1}),$$

then on the basis of Theorem 1 one may assert that the spectrum of system (5) with the additional condition

$$\psi(r_\infty; \lambda) = 0, \quad \text{if } p = 1,$$

and

$$\psi(r; \lambda) \in L_2(0, r_\infty), \quad \text{if } p > 1,$$

will consist of the shortened sequence $\{\lambda_n\}_p^\infty$, and $\{\varphi_{\mathfrak{P}}(r; \lambda_n)\}_p^\infty$ will be the corresponding sequence of fundamental functions. This particular result was previously obtained by Crum ([1]) and served as the impetus for establishing the general Theorem 1.

Let us note, in particular, also the following consequence of Theorem 1 (obtained for a more general choice of the polynomial $\mathfrak{P}$:

$$\mathfrak{P}(\lambda) = (\lambda - \lambda_{k_1})(\lambda - \lambda_{k_2}) \cdots (\lambda - \lambda_{k_p})).$$

The sequence $\{\varphi_n(r)\}_0^\infty$ of fundamental functions of any regular Sturm–Liouville problem on the interval (a, b) always possesses the following property:

Let the integers $(0 \leq) k_1 < k_2 < \cdots < k_p$ be such that the product

$$(k - k_1)(k - k_2) \cdots (k - k_p)$$

is nonnegative for $k = 0, 1, 2, \ldots$; then the determinant

$$W_*(\varphi_{k_1}, \varphi_{k_2}, \ldots, \varphi_{k_p})$$

preserves its sign inside (a, b).

3. Theorem 1, together with previous results ([3]), makes it possible to indicate a simple procedure for reconstructing a differential system from its spectral function τ of the form

$$\tau(\lambda) = \frac{1}{\pi} \int_0^\lambda R(\mu) \frac{d\mu}{\sqrt{\mu}} \quad (\lambda \geq 0); \qquad \tau(\lambda) = 0 \quad (\lambda \leq 0), \tag{6}$$

where $R(\lambda)$ is some rational function, nonnegative on the positive axis. Here we are speaking of reconstructing from the function τ a continuous potential $V(r)$ $(0 \leq r < \infty)$, as well as the constant h in the case

of the system $(S_{0,h})$ (with $r_\infty = \infty$) and the function
$V(r) = p(p-1)/r^2 + q(r)$ $(0 \le r < \infty)$ in the case of the system

$$\psi'' - \left[\frac{p(p-1)}{r^2} + q(r)\right]\psi + \lambda\psi = 0, \qquad \lim_{r\downarrow 0} r^{-p}\psi(r;\lambda) = 1/1{\cdot}3\ldots(2p-1), \qquad (S_p)$$

where $q(r)$ $(0 \le r < \infty)$ is a certain continuous function, and p is a natural number.

In the first case it is necessary that $R(\lambda) = 1 + o(1)$, while in the second $R(\lambda) = \lambda^p + o(\lambda^p)$ as $\lambda \to \infty$. If these necessary conditions are satisfied, the problem will have a unique solution. Since in the second case the function $R(\lambda)$ can always be represented in the form $R(\lambda) = \mathcal{P}_1(\lambda)R_1(\lambda)$, where $\mathcal{P}_1(\lambda)$ is a polynomial of degree p, nonnegative for $\lambda \ge 0$, and $R_1(\lambda)$ is a rational function, finite for $\lambda \ge 0$, such that $R_1(\lambda) = 1 + o(1)$ as $\lambda \to \infty$, it is enough, on the basis of Theorem 1, to give the solution of the problem for the first case. The way of obtaining an effective solution of the problem for this case was indicated already in (3). We give the final formulas.

Let $R(\lambda) = P(\lambda)/Q(\lambda)$, where $P(\lambda)$ and $Q(\lambda)$ are relatively prime polynomials of degree n with leading coefficients equal to one. Since $Q(k^2) > 0$ for real k, we have $Q(k^2) = Q_+(k)Q_-(k)$, where $Q_+(k)$ is a polynomial of degree n with leading coefficient one, having roots only in the upper half-plane $\operatorname{Im} k > 0$, and $Q_-(k) = (-1)^n Q_+(-k) = \overline{Q_+(\bar{k})}$. Put

$$C(r;k) = \frac{i^n}{2}\,[Q_+(k)e^{ikr} + Q_+(-k)e^{-ikr}], \tag{7}$$

$$S(r;k) = \frac{i^n}{2i}\,[Q_+(k)e^{ikr} - Q_+(-k)e^{-ikr}].$$

Assuming further that $P(k^2) = (k^2 - k_1^2)(k^2 - k_2^2)\cdots(k^2 - k_n^2)$, where all k_j^2 $(j = 1, 2, \ldots, n)$ are distinct, one may assert that the spectral function τ under consideration corresponds to the system $(S_{0,h})$ with $r_\infty = \infty$,

$$V(r) = -2\frac{d^2}{dr^2}\ln W(C(r;k_1), \ldots, C(r;k_n)) \tag{8}$$

and h equal to the product by $-2i$ of the sum of all residues inside the upper half-plane of the function $R(k^2) - 1$. It is noteworthy that for the solution φ of this system one can give an explicit formula, namely:

$$\varphi(r;\lambda) = P^{-1}(\lambda)\,\frac{W(C(r;k), C(r;k_1), \ldots, C(r;k_n))}{W(C(r;k_1), \ldots, C(r;k_n))}. \tag{9}$$

It should also be noted that, if in the last expression the functions $C(r;k)$, $C(r;k_j)$ $(j=1,2,\dots,n)$ are replaced by the corresponding functions $S(r;k)$, $S(r;k_j)$ $(j=1,2,\dots,n)$, then instead of the function φ we obtain, multiplied by $k=\sqrt{\lambda}$, the solution ψ of the system (S_p) (for the case $p=1$) with $V(r)=q(r)$, expressed by formula (8) with $C(r;k_j)$ replaced by the corresponding functions $S(r;k_j)$ $(j=1,2,\dots,n)$; in this case the indicated differential system (S_1) will correspond to the spectral function $\tau(\lambda)$ having the following expression:

$$\tau(\lambda)=\frac{1}{\pi}\int_0^\lambda R(\mu)\sqrt{\mu}\,d\mu\quad(\lambda\geq 0);\qquad \tau(\lambda)=0\quad(\lambda\leq 0).$$

Naturally, all the results stated in § 3 are directly generalized to the case when the polynomial $P(\lambda)$ has multiple roots.

4. To simplify the final formulations, we have assumed everywhere that the functions $V(r)$ and $q(r)$ are continuous on $(0,r_\infty]$; however, all our conclusions can easily be reformulated for the case when these functions belong to the class $L_1(0,r_\infty]$, i.e., are summable on each interval $(0,l)$, where $l<r_\infty$.

Theorem 2. In order that a given nondecreasing function $\tau(\lambda)=\tau(\lambda-0)$ $(-\infty<\lambda<\infty;\ \tau(-\infty)=0)$ be the spectral function of some system $(S_{0,h})$ with potential $V(r)\in L_1(0,r_\infty]$, where r_∞ $(0<r_\infty\leq\infty)$ is prescribed, it is necessary and sufficient that: 1) the function

$$\Pi(t)=\int_{-\infty}^{\infty}\frac{1-\cos\sqrt{\lambda}\,t}{\lambda}\,d\tau(\lambda)\qquad(0\leq t<2r_\infty)$$

be finite and have two absolutely continuous derivatives on every segment $(0,l)$ $(l<2r_\infty)$; 2) $\Pi'(0)=1$; 3) $\limsup_{\rho\to\infty}(N(\rho)/\sqrt{\rho})\geq r_\infty/\pi$, where $N(\rho)$ is the number of points of the spectrum of the function τ in the interval $(0,\rho)$.

This theorem was first stated in ([4]) (see Theorem 6); however, in its formulation there, through an oversight, the third condition of Theorem 2 was omitted. Some of the methods needed in its proof are also given in ([5]). Let us also recall that the "transition" function $\Pi(t)$ is uniquely determined by the system $(S_{0,h})$ (it does not depend on the choice of its spectral function), and moreover $\Pi''(0)=-h$, while the function $\Pi'''(2r)$ is always of the same smoothness as the function $V(r)$, $(0\leq r<r_\infty)$. The latter means that if one of the functions $\Pi'''(2r)$, $V(r)$ is continuous or absolutely continuous, or, moreover, has a certain number of continuous (absolutely continuous) derivatives on every segment $(0,l)$ of the half-open interval $(0,r_\infty]$, then the other function will have the same property.

Theorem 3. In order that a nondecreasing function $\tau(\lambda)=\tau(\lambda-0)$ $(-\infty<\lambda<\infty;\ \tau(-\infty)=0)$ be the spectral function of some system (S_p) with continuous function $q(r)$ $(0\leq r<r_\infty)$ and natural p, it is necessary and sufficient that the function

$$\tau_*(\lambda) = \int_{-\infty}^{\lambda} \frac{d\tau(\mu)}{(\mu^2+1)^{p/2}}$$

be the spectral function of some system $(S_{0,h})$ with continuous potential $V(r)$ $(0 \le r < r_\infty)$; moreover, the function $q(r)$ $(0 \le r < r_\infty)$ will always be of the same smoothness as the potential $V(r)$ $(0 \le r < r_\infty)$.

Since the asymptotic behavior of the spectral functions of systems $(S_{0,h})$ has been well studied ([6,7]), Theorem 3 makes it possible to draw a number of conclusions about the asymptotic behavior of the spectral functions of systems (S_p) (in particular, to refine the result obtained for them in ([6])).

Odessa Hydrotechnical Institute

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CITED LITERATURE

[1] M. M. Crum, Quart. J. Math. Oxford (2), 6 (1955).
[2] L. V. Goncharov, *Theory of Approximation and Interpolation of Functions*, 1934.
[3] M. G. Krein, DAN, 94, No. 6 (1954).
[4] M. G. Krein, DAN, 88, No. 3 (1953).
[5] M. G. Krein, DAN, 97, No. 1 (1954)*.
[6] B. M. Levitan, Izv. AN SSSR, ser. matem., 19, No. 1 (1955).
[7] V. A. Marchenko, Izv. AN SSSR, ser. matem., 19, No. 6 (1955).

* The author takes this opportunity to point out that in formula (17) of that article the coefficient 2 is missing before the integral sign.

Note: Figure translations are in progress. See original paper for figures.