MATHEMATICS

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Abstract

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OPERATIONAL CALCULI FOR FUNCTIONS DEFINED ON THE WHOLE LINE

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We shall say that a function $f(x)$ belongs to the set S if:

1) $f(x)$ is defined almost everywhere on the line $-\infty < x < \infty$ and is Lebesgue integrable on every finite interval.
2) There exists at least one pair of values p for which the Laplace integrals

$$\int_a^{\infty} f(x)e^{-p_1 x}\,dx \quad \text{and} \quad \int_{-\infty}^{b} f(x)e^{-p_2 x}\,dx$$

converge. From the convergence of the integrals there follows the existence of numbers σ_1 and σ_2 such that everywhere in the half-plane $\operatorname{Re} p > \sigma_1$ the first integral converges and represents an analytic function, regular in this half-plane. Similarly, the second integral converges everywhere in the half-plane $\operatorname{Re} p < \sigma_2$ and in this half-plane represents an analytic regular function.

Denote by S_+ the set of all functions from S that are equal to zero for all $x < a$. Denote by S_- the set of all functions from S that are equal to zero for $x > b$. The numbers a and b depend on the choice of the function $f(x)$. The Laplace transform

$$\overline{f}_1(p) = \int_{-\infty}^{+\infty} f_1(x)e^{-px}\,dx, \qquad f_1(x) \in S_+, \tag{1}$$

maps the set S_+ into a set of functions of the complex variable $p = \sigma + i\tau$, regular in a right half-plane; we shall denote this set by $\overline{S}_+$.

The transform

$$\overline{f}_2(p) = -\int_{-\infty}^{+\infty} f_2(x)e^{-px}\,dx, \qquad f_2(x) \in S_-,$$

maps the set S_- into a set of functions of the complex variable $p = \sigma + i\tau$, regular in a left half-plane; we shall denote this set by $\overline{S}_-$.

The sets $\overline{S}_+$ and $\overline{S}_-$, with the usual definition of addition and multiplication by a complex number, form linear sets. Denote by $\mathfrak{M}$ the direct sum of the sets $\overline{S}_+$ and $\overline{S}_-$, i.e. the set of all pairs $(\overline{f}_1(p), \overline{f}_2(p))$, $\overline{f}_1(p) \in \overline{S}_+$ and $\overline{f}_2(p) \in \overline{S}_-$. In this case the sum of two pairs $(\overline{f}_1(p), \overline{f}_2(p))$, $(\overline{g}_1(p), \overline{g}_2(p))$ is called the pair $(\overline{f}_1(p) + \overline{g}_1(p), \overline{f}_2(p) + \overline{g}_2(p))$, and the product of $(\overline{f}_1(p), \overline{f}_2(p))$ by a complex number λ is called the pair $(\lambda\overline{f}_1(p), \lambda\overline{f}_2(p))$.

Consider in $\mathfrak{M}$ the linear subset $\mathfrak{M}_0$, consisting of all pairs of the form $(\bar{\theta}(p), \bar{\theta}(p))$, where

$$\bar{\theta}(p) = \int_a^b \theta(x)e^{-px}\,dx;$$

$\theta(x)$ belongs to S_+S_-—the intersection of the sets S_+ and S_-.

Denote by $\bar{S}$ the quotient set $\mathfrak{M}/\mathfrak{M}_0$. The elements of $\bar{S}$ are classes. Two elements of $\mathfrak{M}$ belong to one and the same class if their difference belongs to $\mathfrak{M}_0$. As is known, $\bar{S}$ will be a linear set.

Theorem. *The linear sets S and $\bar{S}$ are isomorphic.*

Let $f(x) \in S$,

$$\bar{f}_+(p) = \int_0^\infty f(x)e^{-px}\,dx, \qquad \bar{f}_-(p) = -\int_{-\infty}^0 f(x)e^{-px}\,dx.$$

To the function $f(x)$ we assign the element of the set $\bar{S}$ whose representative is the pair $(\bar{f}_+(p), \bar{f}_-(p))$, i.e. the coset to which $(\bar{f}_+(p), \bar{f}_-(p))$ belongs; denote this class by $\bar{f}$. The mapping of S into $\bar{S}$ thus constructed will be the desired isomorphism. First of all, it is clear that this mapping is linear; we now prove that every element $\bar{g} \in \bar{S}$ is the image of some element $g(x) \in S$.

Indeed, let $(\bar{g}_1(p), \bar{g}_2(p))$ be a representative of the class $\bar{g}$. By the definition of the pair $(\bar{g}_1(p), \bar{g}_2(p))$, there exist functions $g_1(x) \in S_+$ and $g_2(x) \in S_-$ such that

$$\bar{g}_1(p) = \int_a^\infty g_1(x)e^{-px}\,dx, \qquad \bar{g}_2(p) = -\int_{-\infty}^b g_2(x)e^{-px}\,dx.$$

Recall that $g_1(x) = 0$ for $x < a$ and $g_2(x) = 0$ for $x > b$. The function $g(x) = g_1(x) + g_2(x)$ is the preimage of $\bar{g} \in \bar{S}$. Indeed, if $a < 0$ and $b < 0$, then

$$\bar{g}_+(p) = \int_0^\infty (g_1(x) + g_2(x))e^{-px}\,dx =$$

$$= \int_0^\infty g_1(x)e^{-px}\,dx = \bar{g}_1(p) - \int_a^0 g_1(x)e^{-px}\,dx,$$

$$\bar{g}_-(p) = -\int_{-\infty}^0 (g_1(x) + g_2(x))e^{-px}\,dx =$$

$$= -\int_{-\infty}^b g_2(x)e^{-px}\,dx - \int_a^0 g_1(x)e^{-px}\,dx,$$

$$\bar{g}_-(p) = \bar{g}_2(p) - \int_a^0 g_1(x)e^{-px}\,dx.$$

It follows from this that the pairs $(\bar{g}_+(p), \bar{g}_-(p))$ and $(\bar{g}_1(p), \bar{g}_2(p))$ belong to the same class $\bar{g}$. Similarly one can show that in the other cases ($a > 0,\ b < 0;\ a > 0,\ b > 0;\ a < 0,\ b > 0$) the pairs $(\bar{g}_+(p), \bar{g}_-(p))$ and $(\bar{g}_1(p), \bar{g}_2(p))$ belong to one and the same class.

Thus, the mapping under consideration is a mapping of S onto $\overline{S}$. If the image $h(x)$ of the element $\overline{h}$ is equal to zero almost everywhere, then the class $\overline{h}$ coincides with the set $\mathfrak{M}_0$. Conversely, if $\overline{h}$ coincides with $\mathfrak{M}_0$, then the image of the element $\overline{h}$ is equal to zero almost everywhere. Indeed, as a representative of h one may take the pair $(0, 0)$; consequently,

$$\int_a^\infty h_1(x)e^{-px}\,dx = 0 \quad \text{and} \quad \int_{-\infty}^b h_2(x)e^{-px}\,dx = 0,$$

whence almost everywhere $h_1(x) \equiv 0$ and $h_2(x) \equiv 0$, and therefore $h(x) \equiv 0$. Thus the one-to-one correspondence of the set S onto $\overline{S}$ has been proved.

Let us consider pairs $(F_1(p), F_2(p))$, where the function $F_1(p)$ is equal to the quotient of two functions of the set $\overline{S}_+$, and $F_2(p)$ is equal to the quotient of two functions of the set $\overline{S}_-$. We shall denote the set of all such pairs by $\overline{\mathfrak{M}}$.

If the pair $(F_1(p), F_2(p))$ belongs to $\overline{\mathfrak{M}}$, then there exist in $\mathfrak{M}$ elements $(\overline{f}_1(p), \overline{f}_2(p))$ such that the pairs $(F_1(p)\overline{f}_1(p), F_2(p)\overline{f}_2(p))$ again belong to $\mathfrak{M}$. To each pair $(F_1(p), F_2(p))$ of the set $\overline{\mathfrak{M}}$ we shall put in correspondence an operator F. Denote by $\overline{\Omega}_F$ the totality of all elements $(\overline{f}_1(p), \overline{f}_2(p))$ of the set $\mathfrak{M}$ for which $(F_1(p)\overline{f}_1(p), F_2(p)\overline{f}_2(p))$ again belongs to $\mathfrak{M}$. To the set $\overline{\Omega}_F$ there corresponds a certain subset in the set $\overline{S}$. We shall denote the image of this subset under the isomorphism $S \leftrightarrow \overline{S}$ by Ω_F. Consequently, if $f(x)$ belongs to Ω_F, then among the elements of the class $\overline{f}$ there exist such pairs $(\overline{f}_1(p), \overline{f}_2(p))$ that $(F_1(p)\overline{f}_1(p), F_2(p)\overline{f}_2(p))$ belongs to $\mathfrak{M}$.

On the set Ω_F we define the operator F, putting, for $f \in \Omega_F$,

$$Ff = g;$$

here g is the image of the class $\overline{g}$, whose representative is the pair $(F_1(p)\overline{f}_1(p), F_2(p)\overline{f}_2(p))$. It is clear that the operator F thus defined will be linear, but, generally speaking, not single-valued, i.e. to one and the same function $f(x)$ there may correspond an infinite set of functions Ff. This is explained by the fact that the pairs $(F_1(p)\overline{\theta}(p), F_2(p)\overline{\theta}(p))$ need not again belong to the set $\mathfrak{M}_0$.

By the set of indeterminacy of the operator F we shall mean the image of the set of all pairs $(F_1(p)\overline{\theta}(p), F_2(p)\overline{\theta}(p))$ under the homomorphism of $\mathfrak{M}$ onto S, where $(\theta(p), \theta(p))$ is any pair belonging to $\mathfrak{M}_0\overline{\Omega}_F$.

Obviously, the set of indeterminacy of the operator consists of the values of the operator F over the zero element of the set S. The operator F corresponding to the pair $(F_1(p), F_2(p))$ will be single-valued only in the case when its set of indeterminacy is empty. For this it is necessary and sufficient that the pairs $(F_1(p)\overline{\theta}(p), F_2(p)\overline{\theta}(p))$ belong to the set $\mathfrak{M}_0$ for all $(\overline{\theta}(p), \overline{\theta}(p))$ belonging to $\mathfrak{M}_0\overline{\Omega}_F$.

We shall call the sum of the operators F and G the operator corresponding to the pair $(F_1(p)G_1(p), F_2(p)G_2(p))$, and the product of the operators we shall call the operator corresponding to the pair $(F_1(p)G_1(p), F_2(p)G_2(p))$. We denote the sum of the operators by $F + G$, and the product by FG.

In conclusion we give several examples. In those cases when $F_1(p)$ and $F_2(p)$ can be obtained as analytic continuations of the function $F(p)$, we shall write $F(p)f$ instead of Ff.

1. Operator $\dfrac{1}{p^n}$. In this case

$$\frac{1}{p^n} f = \frac{1}{(n-1)!} \int_0^x (x-\xi)^{n-1} f(\xi)\, d\xi + \sum_{k=0}^{n-1} a_k x^k.$$

The indeterminacy set of the operator $\dfrac{1}{p^n}$ consists of all polynomials of degree not higher than $n-1$.

2. **The operator p^n.** Its domain of definition consists of functions differentiable n times, whose derivative of order n belongs to S, and

$$p^n f(x) = f^{(n)}(x).$$

3. **Regular operators.** Let the function $F(p)$ be regular at the infinitely distant point of the complex p-plane. Then the operator corresponding to

the pair $(F(p), F(p))$ is called regular. Such an operator is defined on the whole set S. The value of the operator can be computed by the formula

$$F(p)f(x) = \sum_{k=0}^{\infty} \frac{a_k}{p^k} f(x) + \frac{1}{2\pi i} \int_{|p|=\rho_1>\rho_0} F(p)\,\bar{\theta}(p) e^{pt}\,dp.$$

Here $\theta(p)$ is an arbitrary function from $\mathfrak{M}_0$ and

$$\sum_{k=0}^{\infty} \frac{a_k}{p^k} = F(p), \qquad |p| > \rho_0.$$

For example, the operator $\dfrac{1}{\sqrt{p^2+\lambda^2}}$ is regular; the value of this operator on $f(x) \equiv 1$ is equal to

$$\frac{1}{\sqrt{p^2+\lambda^2}}\,1 = J_0(\lambda x) + \frac{1}{\pi} \int_{-\lambda}^{\lambda} \frac{\bar{\theta}(i\xi) e^{i\xi x}\,d\xi}{\sqrt{\lambda^2-\xi^2}},$$

where $\bar{\theta}(p)$ is an arbitrary function from $\mathfrak{M}_0$.

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Note: Figure translations are in progress. See original paper for figures.

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