AN ATTEMPT AT COMPUTING A POTENTIAL FUNCTION IN THE LOWER HALF-PLANE FROM ITS VALUES MEASURED ON THE EARTH’ S SURFACE

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Abstract

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AN ATTEMPT AT COMPUTING A POTENTIAL FUNCTION IN THE LOWER HALF-PLANE FROM ITS VALUES MEASURED ON THE EARTH' S SURFACE

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In gravity prospecting and magnetic prospecting, the computation of a potential function in the lower half-plane from its values known on the earth' s surface is usually carried out by means of an approximate formula obtained on the basis of the theorem on the mean value of a harmonic function:

$$u(0,0) \approx \frac{u(\Delta,0)+u(-\Delta,0)+u(0,\Delta)+u(0,-\Delta)}{4}, \quad (1)$$

or else the relation established by Renvoy ([1]) is used:

$$u(0,\Delta) \approx \frac{1}{\pi}\sum_{n=-\infty}^{+\infty} u(n\Delta,0)\,\frac{(-1)^n e^{\pi}-1}{1+n^2}, \quad (2)$$

where Δ is a constant equal to the distance between observation points; n is an integer. Renvoy gives the derivation of formula (2) without any indication of an error estimate.

Relation (2) can be obtained with the aid of the integral (3), which was studied by A. A. Zamorev ([2]):

$$u(0,y) = \frac{1}{\pi}\int_0^A e^{\omega y}\,d\omega \int_{-\infty}^{+\infty} u(x,0)\cos\omega x\,dx + \Delta I, \quad (3)$$

where

$$\Delta I = \frac{1}{\pi}\int_A^{\infty} e^{\omega y}\,d\omega \int_{-\infty}^{+\infty} u(x,0)\cos\omega x\,dx; \quad (4)$$

the x-axis is situated horizontally on the earth' s surface, and the y-axis is directed vertically downward. Under the condition of convergence of the integral

$$\int_{-\infty}^{+\infty} |u(x,0)|\, dx$$

one may change the order of integration in (3):

$$u(0,y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} u(x,0) \left[\frac{y}{x^2+y^2} \left(e^{Ay} \cos Ax - 1 \right) + \right.$$

$$\left. + \frac{x}{x^2+y^2} e^{Ay} \sin Ax \right] dx + \Delta I, \tag{5}$$

after which, putting in (5) $A = \pi/\Delta$, $y = \Delta$, $x = n\Delta$, and using the trapezoidal formula, we obtain expression (2), indicated by Renvoy.

The correction term ΔI, absent in Renboux, for the case of a horizontal circular cylinder producing on the earth' s surface a gravity anomaly

$$u'_y(x,0) = \frac{h}{x^2+h^2},$$

makes it possible to compute the magnitude of the relative error

$$\frac{\Delta u'_y(0,y)}{u'_y(0,y)} = \frac{\Delta I}{u'_y(0,y)} = e^{-(h-y)A} \qquad (y \leq h), \tag{6}$$

where h is the depth of the cylinder axis. Expression (6) shows that for $y/h > 0.5$, $A = \pi/\Delta$, and $y = \Delta$, the relative error $\Delta u'_y/u'_y$ begins to increase sharply; therefore Renboux's formula proves suitable for computing the potential function only far from the body.

To improve existing methods for computing the potential function in the lower half-plane, it is proposed to recalculate not the potential u itself, but a certain function $\Phi(w)$ of the complex expression $w = u + iv$, where u and v are conjugate harmonic functions. The function $\Phi(w)$ is chosen in such a way that the composite function $\Phi[w(\tau)] = \Phi[w(x+iy)]$ is analytic in the region of interest to us. For this purpose one should use the formulas

$$\{\Phi[w(iy)] + \Phi[w(-iy)]\} e^{By^2} = \frac{2}{\pi} \int_0^{\infty} \operatorname{ch} \omega y \, d\omega \int_{-\infty}^{+\infty} \Phi[w(x)] e^{-Bx^2} \cos \omega x \, dx \tag{7}$$

or, if $\Phi(w) = 1/w$,

$$\frac{1}{w(iy)} + \frac{1}{w(-iy)} = \frac{2}{\pi} e^{-By^2} \int_0^\infty \operatorname{ch} \omega y \, d\omega \int_{-\infty}^{+\infty} \frac{e^{-Bx^2}}{w(x)} \cos \omega x \, dx, \tag{8}$$

where $\Phi[w(\tau)]$ is an analytic function, regular in the strip $-a < y < b$ ($a > 0,\ b > 0$) and satisfying, in each strip interior to $-a < y < b$, for every positive ε, the conditions

$$|\Phi[w(\tau)]e^{-B\tau^2}| < Le^{-(\lambda-\varepsilon)x}, \qquad x \to \infty, \qquad \lambda > \varepsilon;$$

$$|\Phi[w(\tau)]e^{-B\tau^2}| < Me^{(\mu-\varepsilon)x}, \qquad x \to -\infty, \qquad \mu > \varepsilon,$$

where L, M, μ, λ, and B are certain fixed positive numbers. The stated conditions and formula (7) can be readily obtained on the basis of a theorem given by E. Titchmarsh ([3], p. 61).

If we restrict ourselves to the consideration of two-dimensional bodies of constant density with a cross-section bounded by a broken line with a finite number of links, then the third derivative $\partial^3 w/\partial x\,\partial y^2$, which can be found from gravimetric survey data, is a rational fraction

$$\frac{\partial^3 w}{\partial x\,\partial y^2} = \frac{P(\tau)}{C(\tau-\tau_1)(\tau-\tau_2)\cdots(\tau-\tau_n)}, \tag{9}$$

where $P(\tau)$ is some polynomial, and $\tau_1, \tau_2, \dots, \tau_n$ are the values of τ at the angular points of the broken line. Taking in formula (8), as w, the third derivative (9) and decomposing

$$\frac{1}{\partial^3 w/\partial x\,\partial y^2}$$

into a polynomial and simple fractions, one can verify formula (8) directly. Under our assumptions, the special points of

$$\frac{1}{\partial^3 w/\partial x\,\partial y^2}$$

will be the roots $\alpha_k + i\beta_k$ of the polynomial $P(\tau)$, and formula (8) turns out—

is found to be valid for those points (x, y) whose ordinates satisfy the inequality $y^2 < \alpha_k^2 + \beta_k^2$. In cases practically important for gravitational prospecting, the roots of the polynomial $P(\tau)$ are arranged in such a way that they do not interfere with the determination of the positions of the most interesting nodal points, or are altogether absent, as occurs, for example, for a bed or a step.

Fig. 1. Example of continuation into the lower half-plane. The numbers by the isolines have been changed by an arbitrary factor.

Figure 1: Fig. 1. Example of continuation into the lower half-plane. The numbers by the isolines have been changed by an arbitrary factor.

For $A = \pi/\Delta,\ y = \Delta,\ B = 0.1/\Delta^2$, using the techniques already applied to transform expression (3), we reduce formula (7) to a form convenient for computation:

$$\Phi[w(i\Delta)] + \Phi[w(-i\Delta)] \simeq \frac{1}{\pi} \sum_{n=-\infty}^{+\infty} \Phi[w(n\Delta)] \frac{(-1)^n e^{\pi - 0.1(n^2+1)}}{1+n^2} + \Delta I_1 =$$

$$= k_0 \Phi[w(0)] + \sum_{n=1}^{\infty} k_n \{\Phi[w(n\Delta)] + \Phi[w(-n\Delta)]\} + \Delta I_1, \tag{10}$$

where

$$\Delta I_1 = \frac{1}{\pi} e^{-By^2} \int_A^{\infty} e^{\omega y}\, d\omega \int_{-\infty}^{+\infty} \Phi[w(x)] e^{-Bx^2} \cos \omega x\, dx;$$

$$k_0 = 6.665; \qquad k_1 = -3.015; \qquad k_2 = 0.8935; \qquad k_3 = -0.2710;$$

$$k_4 = 0.07918; \quad k_5 = -0.02104; \quad k_6 = 0.004921; \quad k_7 = -0.0009929.$$

Fig. 1. Example of recalculation into the lower half-plane. The numbers by the isolines have been changed by an arbitrary number of times

If we introduce the function $u_1(x, y) = \frac{1}{2}[u(x, y) + u(-x, y) + u(x, -y) + u(-x, -y)]$, for which $\partial u_1(x, 0)/\partial y = 0,\ \partial u_1(0, y)/\partial x = 0$, and use the Cauchy-Riemann equations, then for u_1 one may find a conjugate function v_1 satisfying the relations $v_1(x, 0) = 0,\ v_1(0, y) = 0$. Using these relations, formula (10) can be given the following form:

$$2\Phi[u(0, \Delta) + u(0, -\Delta)] \simeq$$

$$\simeq k_0 \Phi[2u(0, 0)] + \sum_{n=1}^{\infty} 2k_n \Phi[u(n\Delta, 0) + u(-n\Delta, 0)] + \Delta I_1. \tag{11}$$

Using the approximate equality (1), one can obtain relations analogous to formulas (10) and (11):

$$\Phi[w(0)] \simeq \frac{\Phi[w(\Delta)] + \Phi[w(-\Delta)] + \Phi[w(i\Delta)] + \Phi[w(-i\Delta)]}{4}, \qquad (12)$$

$$\Phi[2u(0,0)] \simeq \frac{\Phi[u(\Delta,0) + u(-\Delta,0)] + \Phi[u(0,\Delta) + u(0,-\Delta)]}{2}. \qquad (13)$$

Let us note that, although when using formulas (11) and (13) the amount of computational work is much smaller than for formulas (10) and (12), nevertheless formulas (10) and (12) differ favorably from expressions (11) and (13) by the advantageous location of the zeros of the polynomial $P(\tau)$.

In Fig. 1 are shown the curves of the horizontal and vertical components of the magnetic-field intensity, H and Z, measured on the surface of the earth above a suite of ferruginous quartzites in the region of the Kursk magnetic anomaly. From the values of H and Z, by means of the Poisson integral, the derivatives $\partial H/\partial x$ and $\partial Z/\partial x$ were found at points of the upper half-plane[4]. Taking $w = \partial H/\partial x - i\partial Z/\partial x$, by formula (12) the values of $\partial H/\partial x$ and $\partial Z/\partial x$ were computed at points of the lower half-plane. The distribution pattern of the isolines $(\partial H/\partial x)^2 + (\partial Z/\partial x)^2$ is shown in the lower part of Fig. 1. Points A and B, located inside closed isolines and corresponding to the corner points of the bed, occur at a depth of 180 m. The actual depth of occurrence of the upper edge of the ferruginous quartzites, according to drilling data, is 157-173 m.

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Note: Figure translations are in progress. See original paper for figures.