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A. V. Efimov

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Abstract

Full Text

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A. V. Efimov

ON THE APPROXIMATION OF CERTAIN CLASSES OF CONTINUOUS FUNCTIONS BY FOURIER SUMS AND FEJÉR SUMS

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Let $\mathfrak{M}$ be some class of continuous functions and let U_n be a linear operator (method of approximation) assigning to each function $f \in \mathfrak{M}$ some polynomial of order n, whose value at the point x we denote by $U_n(f, x)$.

We consider two problems:

I. To find the principal term of the deviation of the function $f(x)$ from $U_n(f, x)$, with a remainder term uniform with respect to the whole class $\mathfrak{M}$, i.e., to obtain a representation:

$$f(x) - U_n(f, x) = AU_n(f, x) + O(BU_n(\mathfrak{M})).$$

II. To investigate the behavior of the upper bound

$$\mathcal{E}_{U_n}(\mathfrak{M}) = \sup_{f \in \mathfrak{M}} \|f(x) - U_n(f, x)\| = \sup_{f \in \mathfrak{M}} \max_{x} |f(x) - U_n(f, x)|,$$

i.e., the upper bound of the deviations of the function $f(x)$ from $U_n(f, x)$, extended over the whole class $\mathfrak{M}$.

In a number of cases the solution of problem I makes it possible to find an asymptotically exact solution of problem II. Indeed, we have

$$\mathcal{E}_{U_n}(\mathfrak{M}) = \sup_{f \in \mathfrak{M}} \|AU_n(f, x)\| + O(BU_n(\mathfrak{M})).$$

Therefore, if

$$BU_n(\mathfrak{M}) = o(\mathcal{E}_{U_n}(\mathfrak{M})),$$

then

$$\mathcal{E}_{U_n}(\mathfrak{M}) \asymp \sup_{f\in\mathfrak{M}} \|AU_n(f,x)\|.$$

The first problem was first solved by E. V. Voronovskaya (^{1}) for the approximation of twice differentiable functions by S. N. Bernstein polynomials. The order of decrease of $\mathcal{E}_{U_n}(\mathfrak{M})$ for certain important classes of functions and methods of approximation was given by Lebesgue (^{2}), Jackson (^{3}), and S. N. Bernstein (^{4}) in 1910-1912. The method of obtaining an asymptotically exact equality from the expression for the principal term of the deviation was used earlier by S. B. Stechkin (^{5}) for approximations of certain classes of analytic functions by Taylor sums.

Let us define some classes of continuous functions of period 2π. For any integer $k \geq 0$ put

$$\omega_k(h,f) = \sup_{|\delta|\leq h} \|\Delta_\delta^k f(x)\|,$$

where

$$\Delta_\delta^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f\left(x + (k-2i)\frac{\delta}{2}\right), \qquad k \geqslant 0 \quad \left(\Delta_\delta^0 f(x) = f(x)\right).$$

We shall say that the function $f(x)$ belongs to the class $MW^rH_k^\alpha$ if its Weyl derivative $f^{(r)}(x)$ of order $r \geqslant 0$ $\left(f^{(0)}(x) = f(x)\right)$ satisfies the condition

$$\omega_k(h, f^{(r)}) \leqslant Mh^\alpha, \qquad 0 < \alpha \leqslant 1.$$

For brevity we shall write MW^r if $k = 0$, and MH_k^α if $r = 0$ and $k \geqslant 1$. The classes MH_2^α were introduced by A. Zygmund (^{6}). The conjugate classes will be denoted respectively by $M\overline{W}^rH_k^\alpha$ for $r > 0$, and by $M\overline{H}_k^\alpha$ for $r = 0$. In addition, by analogy with the classes MW_β^r, introduced in the work of S. B. Stechkin (^{7}), we introduce the classes $MW_\beta^rH_k^\alpha$. We shall say that $f(x) \in MW_\beta^rH_k^\alpha$, $r \geqslant 0,\ 0 < \alpha \leqslant 1$, if $f(x)$ can be represented in the form

$$f(x) = \sum_{m=1}^{\infty} \frac{1}{\pi m^r} \int_{-\pi}^{\pi} \varphi(x+t)\cos\left(mt + \frac{\beta\pi}{2}\right) dt, \tag{1}$$

where $\varphi(x) \in MH_k^\alpha$ $(0 < \alpha \leqslant 1)$ and

$$\int_{-\pi}^{\pi} \varphi(x)\,dx = 0.$$

For $\beta = r$ and $\beta = r+1$ we obtain, respectively, the classes $MW^rH_k^\alpha$ and $M\overline{W}^rH_k^\alpha$. For brevity we shall omit the constant $M = 1$. We shall consider the following methods of approximation: partial Fourier sums, i.e.

$$U_n(f,x) = S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \qquad (n = 0, 1, 2, ...),$$

and Fejér sums, i.e.

$$U_n(f,x) = \sigma_n(f,x) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(f,x).$$

Problem I for the classes H_1^1 and H_2^1 and approximation by Fejér sums was solved by Zamanskii (8).

The order of the quantities $\mathcal{E}_{\sigma_n}(H_1^\alpha)$ was given by S. N. Bernstein (4), while the order of the quantities $\mathcal{E}_{\sigma_n}(\overline{H}_1^\alpha)$ for $0 < \alpha < 1$ follows from results of I. I. Privalov (9) and S. N. Bernstein (4), and the order of $\mathcal{E}_{\sigma_n}(\overline{H}_1)$ was given by Aleksich (10). Asymptotically exact results for $\mathcal{E}_{\sigma_n}(H_1^\alpha)$ $(0 < \alpha \leqslant 1)$ were given by S. M. Nikol' skii (11). The first asymptotically exact result for $\mathcal{E}_{S_n}(\mathfrak{M})$ belongs to A. N. Kolmogorov (12), who considered $\mathcal{E}_{S_n}(W^r)$ in the case of integer $r \geqslant 1$. V. T. Pinkevich (13) considered $\mathcal{E}_{S_n}(W^r)$ for arbitrary $r > 0$. The most general result in this direction belongs to S. M. Nikol' skii (14,15), who proved that for arbitrary $r \geqslant 0$ and $0 < \alpha \leqslant 1$

$$\left.\begin{array}{l} \mathcal{E}_{S_n}\left(W^rH_1^\alpha\right) \\ \mathcal{E}_{S_n}\left(W^r\overline{H}_1^\alpha\right) \end{array}\right\} = \frac{C_1(\alpha)}{\pi}\frac{\ln n}{n^{r+\alpha}} + O\left(\frac{1}{n^{r+\alpha}}\right), \tag{2}$$

where

$$C_1(\alpha) = \sup_{f\in H_1^\alpha}|a_1(f)| = \sup_{f\in H_1^\alpha}\left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx\right| = \frac{2^{1+\alpha}}{\pi}\int_0^{\pi/2} t^\alpha \sin t\,dt.$$

Theorem 1. Let $f(x)$ be a continuous function of period 2π, and let $\omega_2(h,f)$ be its modulus of smoothness. Then

$$f(x) - \sigma_{n-1}(f,x) = -\frac{1}{2\pi}\int_a^\infty \frac{f(x+\frac{2t}{n}) - 2f(x) + f(x-\frac{2t}{n})}{t^2}\,dt + O\left(\omega_2\left(\frac{1}{n}, f\right)\right),$$

where $a > 0$ is an arbitrary constant.

For $\omega_2(h, f) = h$ we obtain Zamansky' s result (8).

Theorem 2. Let $f(x) \in H_2^1$, and let $\overline{f}(x)$ be the conjugate function. Then

$$\overline{f}(x) - \overline{\sigma}_{n-1}(f, x) = -\frac{1}{\pi} \int_0^{a_1} \left[f\left(x + \frac{t}{n}\right) - f\left(x - \frac{t}{n}\right) \right] \frac{\sin t}{t^2}\, dt + O\left(\frac{1}{n}\right),$$

where a_1 is the smallest root of the equation

$$\int_0^u \frac{\sin t}{t}\, dt = \frac{\pi}{2}.$$

From Theorem 2 and the author' s results (17) it follows that:

Theorem 3. The following asymptotic equality holds:

$$\mathcal{E}_{\sigma_n}\left(\overline{H}_2^1\right) = \sup_{f \in H_2^1} \|\overline{f}(x) - \overline{\sigma}_n(f, x)\| = \frac{1}{2\ln(\sqrt{2}+1)} \frac{\ln n}{n} + O\left(\frac{1}{n}\right).$$

Denote by $R_n(f, x)$ the remainder of the Fourier series (1) for a function $f(x) \in W_\beta^r H_k^\alpha$ when $r > 0$, and by $r_{n,\beta}(\varphi, x)$ the remainder of the Fourier series for a function $f(x) \in W_\beta^0 H_k^\alpha$, i.e.,

$$r_{n,\beta}(\varphi, x) = \cos\frac{\beta\pi}{2}\,[\varphi(x) - S_n(\varphi, x)] + \sin\frac{\beta\pi}{2}\,[\overline{\varphi}(x) - \overline{S}_n(\varphi, x)] =$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} [\varphi(x) - \varphi(x+t)] \frac{\sin\left(\frac{2n+1}{2}t + \frac{\beta\pi}{2}\right)}{2\sin\frac{t}{2}}\, dt,$$

where $S_n(\varphi, x)$ and $\overline{S}_n(\varphi, x)$ are the partial sums, respectively, of the Fourier series of the function $\varphi(x)$ and of the conjugate series.

From the results of S. B. Stechkin (16) it follows that, for $0 < \alpha \leq 1$ and $k > 2$, the inclusion

$$C_1 H_k^\alpha \subset H_2^\alpha \subset C_2 H_k^\alpha$$

holds, where $0 < C_1 < C_2$ are certain constants. Hence, using Theorems 1 and 2, we obtain the following theorem.

Theorem 4. Let $f(x) \in W_\beta^r H_k^\alpha$. Then, for any $r > 0$ and $0 < \alpha \leq 1$, the equality

$$R_n(f,x) = f(x) - \sum_{m=1}^{n} \frac{1}{\pi m^r} \int_{-\pi}^{\pi} \varphi(x+t) \cos\left(mt + \frac{\beta\pi}{2}\right)\, dt =$$

$$= \frac{1}{(n+1)^r} r_{n,\beta}(\varphi, x) + O\left(\frac{1}{n^{r+\alpha}}\right).$$

Putting here $\beta = r$ and $\beta = r+1$, we obtain the solution of Problem I for the approximation of functions of the classes $W^r H_k^\alpha$ and $\overline{W}^r H_k^\alpha$ by Fourier sums.

Theorem 5. Let $f(x) \in W_\beta^0 H_k^\alpha$. Then for any $0 < \alpha \leqslant 1$

$$r_{n,\beta}(\varphi, x) = \frac{1}{2\pi^2} \sum_{m=1}^{[\frac{n}{2}]-3} \frac{1}{m^2} \left\{ \int_0^{2m\pi} \left[\varphi(x) - \varphi\left(x + \frac{t}{n} + \frac{5-\beta}{2n}\pi\right)\right] \cos t\, dt - \right.$$

$$\left. - \int_{-2m\pi}^{0} \left[\varphi(x) - \varphi\left(x + \frac{t}{n} - \frac{3+\beta}{2n}\pi\right)\right] \cos t\, dt \right\} + O\left(\frac{1}{n^\alpha}\right).$$

Put

$$C_k(\alpha) = \sup_{f \in H_k^\alpha} |a_1(f)| = \sup_{f \in H_k^\alpha} \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos x\, dx \right|.$$

Theorem 6. For any $0 < \alpha \leqslant 1$ the asymptotic equality

$$\mathcal{E}_{S_n}\left(W_\beta^0 H_k^\alpha\right) = \frac{C_k(\alpha)}{\pi} \frac{\ln n}{n^\alpha} + O\left(\frac{\ln\ln n}{n^\alpha}\right)$$

holds.

From Theorems 4 and 6 there follows

Theorem 7. For any $r \geqslant 0$ and $0 < \alpha \leqslant 1$ the asymptotic equality

$$\mathcal{E}_{S_n}\left(W_\beta^r H_k^\alpha\right) = \frac{C_k(\alpha)}{\pi} \frac{\ln n}{n^{r+\alpha}} + O\left(\frac{\ln\ln n}{n^{r+\alpha}}\right)$$

holds.

Putting in Theorems 5–7 $\beta = r$ and $\beta = r+1$, we obtain the solution of problems I and II for the approximation of functions of the classes H_k^α and $\overline{H}_k^\alpha$ by Fourier sums, and the solution of problem II for the approximation of functions of the classes $W^r H_k^\alpha$ and $W^r \overline{H}_k^\alpha$ by Fourier sums.

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Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

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REFERENCES

1. E. V. Voronovskaya, DAN, 79 (1932).
2. H. Lebesgue, Bull. Soc. Math. de France, 38, 184 (1910).
3. D. Jackson, Über die Genauigkeit der Annäherung stetiger Funktionen durch ganze rationale Funktionen, Göttingen, 1911.
4. S. N. Bernstein, Soobshch. Kharkovsk. matem. obshch., ser. 2, 13, 49 (1912).
5. S. B. Stechkin, Izv. AN SSSR, ser. matem., 17, 461 (1953).
6. A. Zygmund, Duke Math. J., 12, 47 (1945).
7. S. B. Stechkin, Izv. AN SSSR, ser. matem., 20, 643 (1956).
8. M. Zamansky, Ann. Sci. Ecole Norm. Sup., 66, 19 (1949).
9. I. Privaloff, Bull. Soc. Math. de France, 44, 100 (1916).
10. G. Alexits, Matematikai es Fizikai Lapok, 48, 410 (1941).
11. S. M. Nikol' skii, Izv. AN SSSR, ser. matem., 4, 501 (1940).
12. A. Kolmogoroff, Ann. of Math., 36, 521 (1935).
13. V. T. Pinkevich, Izv. AN SSSR, ser. matem., 4, 521 (1940).
14. S. M. Nikol' skii, DAN, 32, 386 (1941).
15. S. M. Nikol'skii, Tr. Matem. inst. im. V. A. Steklova AN SSSR, 15 (1945).
16. S. B. Stechkin, Izv. AN SSSR, ser. matem., 15, 219 (1951).
17. A. V. Efimov, Izv. AN SSSR, ser. matem., 21, 283 (1957).