MATHEMATICS

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1957

SovietRxiv

Abstract

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ON CLASSES OF MODELS WITH A GENERATION OPERATION

In the theory of algebras the concept of generating elements plays an essential role. Transferring this concept to the theory of models, it is natural to say that a set of elements S of some model M of a given class K generates in M its K-submodel N, if N is the intersection of all K-submodels of the model M that contain the elements S (for the terminology see (1,2)). Since the concept of generation will also be used in another sense, we shall agree to call the indicated generations **natural**. Correspondingly, we shall call a class of models K a **class with natural generations** if the intersection of any system of K-submodels of an arbitrary K-model M is either empty or is a K-submodel of the model M. Below we study certain properties of classes of models with natural generations, and also consider another kind of generation, which arises naturally in the study of algebras with partial operations.

All the classes of models under consideration will be assumed to be abstract. The number of basic predicates and individual symbols in these classes may be infinite, but each predicate is assumed to depend on a finite number of arguments. We shall agree to call a class of models K **pseudoaxiomatizable** if it has the following two properties: 1) if every finite part of some system of axioms of the first-order predicate calculus, written with the aid of the predicate and individual symbols of the class K, is satisfied in a suitable K-model, then the whole system is satisfied in a suitable K-model; 2) for every cardinal number m there exists a cardinal number $\mathfrak{n}(m)$ such that in every K-model containing some system of elements S of cardinality m, there exists a K-submodel of cardinality $\mathfrak{n}(m)$ containing the elements S. According to the local theorem of the first-order predicate calculus, all axiomatizable classes of models are pseudoaxiomatizable. The classes considered by Tarski (2), consisting of models of axiomatizable classes on which part of the predicates is discarded (PC_Δ-classes in Tarski's notation), are also pseudoaxiomatizable. In both named cases $m = \mathfrak{n}$ for sufficiently large m. The totality of all ordered sets of limit cardinalities may serve as an example of a pseudoaxiomatizable class that does not admit axiomatization and is not a PC_Δ-class.

No. 1. Let K be a pseudoaxiomatizable class of models with natural generations; let A be an arbitrary K-model with natural generators a_α, $\alpha \in \Gamma_1$, and let a_λ, $\lambda \in \Gamma_2$, be the remaining elements of A. Denote by $D(a_\alpha, \dots, a_\gamma;\ a_\lambda, \dots, a_\nu)$ the diagram (^{3}) of the submodel C formed in A by the elements $a_\alpha, \dots, a_\gamma, a_\lambda, \dots, a_\nu$, i.e. the set of formulas of the form $P_t(a_i, \dots, a_k)$ or $\sim P_t(a_i, \dots, a_k)$ that are true in C, where P_t are the basic predicate symbols of the class K. To each element $a_\lambda,\ \lambda \in \Gamma_2$, we assign individual constants $x_\lambda,\ y_\lambda$, not belonging to the set of basic constants of the class K, and consider the system of axioms

$$\mathfrak{T} = \{D(a_\alpha, \dots;\ x_\lambda, \dots),\ D(a_\alpha, \dots;\ y_\lambda, \dots),\ y_\xi \neq x_\mu\},$$

where ξ is arbitrary–

free fixed index ranges over Γ_2, and μ ranges over Γ_2. If every finite part of the system $\mathfrak{T}$ were realized on a suitable K-model, then there would be a K-model M in which all axioms of $\mathfrak{T}$ were realized. But then in M there would be two minimal K-models containing the elements a_α, which is impossible. Therefore there are finite sets $\alpha_1, \dots, \alpha_s$ and $\lambda_1, \dots, \lambda_t$ for which in K we shall have

$$\begin{gathered} D'_\xi(a_{\alpha_1}, \dots, a_{\alpha_s}, x_{\lambda_1}, \dots, x_{\lambda_t}) \& D'_\xi(a_{\alpha_1}, \dots, a_{\alpha_s}, y_{\lambda_1}, \dots, y_{\lambda_t}) \to \\ \to y_\xi = x_{\lambda_1} \vee \dots \vee y_\xi = x_{\lambda_t}, \end{gathered} \tag{1}$$

where D'_ξ is a suitable finite subdiagram $D(a_{\alpha_1}, \dots, a_{\alpha_s}, a_{\lambda_1}, \dots, a_{\lambda_t})$. In formula (1) the constants a_i, x_j, y_j are not contained in the set of basic constants of the class K. Therefore expression (1) may be understood as a universal axiom with universal quantifiers over a_i, x_j, y_j.

Theorem 1. *Let, in the pseudoaxiomatizable class of models K with natural generations, the K-model A be generated by the elements a_α. If G is the group of automorphisms of A leaving fixed every element A, then for any $a \in A$ the set aG is finite.*

Let in the diagram $D(A)$ we have $a = a_\xi$. In A the expression $D'_\xi(a_{\alpha_1}, \dots, \dots, a_{\alpha_s}, a_{\lambda_1}, \dots, a_{\lambda_t})$ is true. Therefore the expression $D'_\xi(a_{\alpha_1}, \dots, \dots, a_{\alpha_s}, a_{\lambda_1}g, \dots, a_{\lambda_t}g)$ is also true for any $g \in G$. By virtue of (1), for fixed $\lambda_1, \dots, \lambda_t$, for any $g \in G$ we have

$$ag = a_{\lambda_1} \vee \dots \vee ag = a_{\lambda_t},$$

i.e. $aG \subseteq \{a_{\lambda_1}, \dots, a_{\lambda_t}\}$.

In the class of all algebras of fixed type there holds the stronger equality $aG = a$. However, for models the case $aG \neq a$ is also possible.

It occurs, for example, for the class of all partially ordered sets in which each element has two and only two immediate successors.

No. 2. Let K be a pseudoaxiomatizable class of models; let $\{a_\alpha\}$ be some set of symbols distinct from the individual symbols of the class K. Consider a mapping $a_\alpha \to a_\alpha^\sigma$ of this set into some K-model A_σ such that the a_α^σ are generators of A_σ. Let D_σ be the diagram of the model A_σ, and let $D_{\sigma\nu}(a_{\alpha_1}, \dots, a_{\alpha_p}, u_{\lambda_1}, \dots, u_{\lambda_q})$

be all its finite subdiagrams, where a_α, a_λ are elements from the submodel corresponding to $D_{\sigma\nu}$. In A_τ the axioms

$$(\exists x_1) \dots (\exists x_q)\, D_{\sigma\nu}(a_{\alpha_1}, \dots, a_{\alpha_p}, x_1, \dots, x_q), \tag{2}$$

hold; we shall denote them briefly by $E_{\sigma\nu}(a_{\alpha_1}, \dots, a_{\alpha_p})$.

Lemma. *If in an arbitrary model M (not necessarily from the class K) all axioms (1), considered as universal ones, are satisfied, and for the set $\{a_\alpha\}$ of individual symbols the axioms (2) are satisfied, then M contains a submodel canonically isomorphic to the model A_σ.*

By hypothesis, M contains elements denoted by the symbols a_α, and for each ξ in M there are elements $m_1^\xi, \dots, m_t^\xi$ for which the formula

$$D'_\xi(a_{\alpha_1}, \dots, a_{\alpha_s}, m_1^\xi, \dots, m_t^\xi)$$

is true. With each element u_λ of the model A_σ we associate a new symbol v_λ and consider the auxiliary system $\mathfrak{N}$, formed by the axioms

$$D_{\sigma\nu}(a_{\alpha_1}, \dots, a_{\alpha_s}, v_{\lambda_1}, \dots, v_{\lambda_t})$$

for all possible ν, and also by the axioms

$$v_\xi = m_1^\xi \vee \dots \vee v_\xi = m_t^\xi$$

and the diagram of the model M, the elements a_α in the system $\mathfrak{N}$ being replaced by the corresponding elements of the model M. From the validity in M of axioms (1), (2)

it follows that every finite part of the system $\mathfrak{R}$ is consistent. Therefore the system $\mathfrak{R}$ is consistent as well, asserting that M contains a submodel isomorphic to A_σ.

The lemma is proved. With the aid of this lemma the following main theorem can be proved.

Theorem 2. *Every pseudo-axiomatizable class K of models with natural generations is axiomatizable, and moreover axiomatizable by means only of axioms of Skolem form.*

Let $\mathfrak{S}$ be the totality of all axioms of the form
$(x_1) \dots (x_m)(\exists y_1) \dots (\exists y_n)\mathfrak{B}$, true on all models of the class K.

It is necessary to prove that every model A on which all axioms of the system $\mathfrak{S}$ are true belongs to K. Denote by a_α ($\alpha \in \Gamma$) all elements of A, and with each a_α associate a new symbol c_α. Consider all possible mappings $c_\alpha \to c_\alpha^\sigma$ of the totality $\{c_\alpha\}$ into K-models M_σ of cardinality $\mathfrak{n}(\mathfrak{m})$, where $\mathfrak{m}$ is the cardinality of A. Let D_σ be the diagram of the submodel A_σ, generated within M_σ by the elements c_α^σ. Under any mapping $c_\alpha \to b_\alpha$ of the elements c_α into an arbitrary K-model B, in B the axiom is satisfied

$$\vee_\sigma \&_\nu E_{\sigma\nu}(c_{\alpha_1}, \dots, c_{\alpha_p}) = \& \vee_\sigma E_{\sigma\nu_\sigma}(c_{\alpha_1}, \dots, c_{\alpha_p}), \tag{3}$$

and therefore in K each of the axioms
$\vee_\sigma E_{\sigma\nu_\sigma}(c_{\alpha_1}, \dots, c_{\alpha_p})$ holds, where ν_σ is an arbitrary function carrying out a choice of one term from each member of the first disjunction. In view of the pseudo-axiomatizability of K, it follows from this that there exists a finite set $\{\sigma_1, \dots, \sigma_k\}$ for which the axiom
$E_{\sigma_1\nu_1}(c) \vee \dots \vee E_{\sigma_k\nu_k}(c)$ is true in K, and hence the axiom $(c_1) \dots (c_m)(E_{\sigma_1\nu_1}(c) \vee \dots \vee E_{\sigma_k\nu_k}(c))$, which has Skolem form, is also true. But all Skolem axioms that hold in K belong to $\mathfrak{S}$ and therefore are true on A. Consequently, axiom (3) is true on A, and therefore for some σ the axiom $\&_\nu E_{\sigma\nu}(c_{\alpha_1}, \dots, c_{\alpha_p})$ must hold on A. Axioms (1) hold on K and have Skolem form. Therefore (1) are true on A. According to the lemma it follows hence that the model A has a submodel containing the elements a_α and isomorphic to the K-model A_σ. But the elements a_α exhaust the whole model A, which thus proves to be isomorphic to A_σ.

Corollary. *If in the models of an axiomatizable class K nonempty intersections of K-submodels are K-submodels, then every model M, any finite set of elements of which is contained in a suitable K-submodel, is a K-model.*

This follows immediately from the fact that K can be characterized by axioms of Skolem form (cf. (4)).

No. 3. Theorem 2 does not contain an explicit description of the axioms by means of which one can specify classes with natural generations, and only such classes. The simplest sufficient condition can be formulated as follows. An axiom of explicitly functional type shall mean the conjunction of the following two expressions:

$$(x_1) \dots (x_m)(\exists y_1) \dots (\exists y_n)\mathfrak{A}(x_1, \dots, x_m, y_1, \dots, y_n),$$

$$\mathfrak{A}(x_1, \dots, x_m, y_1, \dots, y_n) \;\&\; \mathfrak{A}(x_1, \dots, x_m, z_1, \dots, z_n) \to\to y_1 = z_1 \;\& \dots \&\; y_n = z_n,$$

where $\mathfrak{A}(\mathfrak{x}, \mathfrak{y})$ contains no quantifiers, $\mathfrak{x} = (x_1, \dots, x_m)$.

Theorem 3. *A class of models K, characterized by axioms of explicitly functional form and universal axioms, has natural generations. An automorphism of a K-model that leaves in place all elements of some generating system leaves in place all elements of the model.*

The proof is the same as in the case of algebras. In form, axioms that are more general than explicitly functional ones are represented by conjunctions of expressions

$$(x_1)(\exists y_1)'(x_2)(\exists y_2)\cdots(x_k)(\exists y_k)\mathfrak{A}(x_1,\dots,x_k,\ y_1,\dots,y_k), \tag{4}$$

$$\begin{gathered}\mathfrak{A}(x_1,\dots,x_k,\ y_1,\dots,y_k)\ \&\ \mathfrak{A}(x_1,\dots,x_k,\ z_1,\dots,z_k)\rightarrow\\ \rightarrow y_1=z_1\ \&\ \cdots\ \&\ y_k=z_k.\end{gathered} \tag{5}$$

However, as is easy to verify, every such conjunction is equivalent to an explicitly functional axiom and several universal axioms. Therefore the conclusion of Theorem 3 also holds for classes characterized by pairs of axioms of the form (4), (5).

No. 4. Generations considered in topological algebras and in algebras with partial operations are not always natural. In connection with this we introduce the following definitions. Suppose models are considered whose basic predicates are numbered by elements of a set Π, and let π be some part of Π. We shall call a submodel N of a model M of the type under consideration π-closed in M if, from the truth of the expression $P_\alpha(x_1,\dots,x_{n_\alpha},y)$, for $\alpha\in\pi$, $\{x_1,\dots,x_{n_\alpha}\}\subseteq N$, $y\in M$, it follows that $y\in N$. It is easy to see that the intersection of an arbitrary system of π-closed submodels is either empty or is a π-closed submodel. Suppose a class of models K is considered. In this class, a submodel N of some K-model M will be called **π-generated by the elements** $\{a_\alpha\}$ if N is the least π-closed K-submodel of the model M containing the elements a_α. We shall call the class K a **class with π-generations** if nonempty intersections of π-closed K-submodels in K-models are K-submodels.

Theorem 4. *If some model M satisfies an axiom of the form*

$$(Q_1x_1)\cdots(Q_{mx}m)\mathfrak{A}(x_1,\dots,x_m),$$

where $\mathfrak{A}(x_1,\dots,x_m)$ is an open positive formula in which the object variables bound in the quantifier part by existential signs occur only in the last places of elementary terms $P_\alpha(x_i,\dots,x_l)$ and only for $\alpha\in\pi$, then the indicated axiom is also satisfied on every π-closed submodel of the model M.

From this, in particular, it follows that every class of models admitting an axiomatization by means of universal axioms and axioms of the form indicated in Theorem 4 will be a class with π-generations.

Received
15.VI.1957

CITED LITERATURE

1. A. I. Mal' tsev, DAN, 109, No. 2, 264 (1956).
2. A. Tarski, Proc. Acad. v. Wetensch., 57, No. 5, 572 (1954).

3. A. Robinson, Complete Theories, Amsterdam, 1956.

4. J. Łoś, R. Suszko, Bull. Acad. Polon. Sci., 3, Cl. 3, No. 4, 201 (1955).

Note: Figure translations are in progress. See original paper for figures.

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