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Abstract

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ON COMPLETE SYSTEMS AND BASES IN L_2

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The article considers questions connected with the problem of completeness in L_2 of systems of functions that are "close" in a certain sense (L_2, as usual, is the Hilbert space of functions square-summable on the segment $[a, b]$). First, a theorem of a general character is proved, which is subsequently used as a criterion in solving the question of the completeness of "close" systems.

§ 1. Let the set $G \subset L_2$ be everywhere dense in L_2. We shall assume that G forms a space of type (B), and moreover that from

$$\lim_{n\to\infty} \|f - f_n\|_G = 0$$

it always follows that

$$\lim_{n\to\infty} \|f - f_n\|_{L_2} = 0$$

(the norm in G in the case $G \neq L_2$ is distinct from the norm in L_2).

The following almost obvious assertion holds:

Theorem 1. *Let a bounded linear operator T be defined on G, mapping G into itself and possessing a unique inverse T^{-1}. If $\{g_n\}$, $n = 1, 2, \ldots$, is a complete system or a basis in G, then the system of functions*

$$f_n = T g_n, \qquad n = 1, 2, \ldots,$$

also forms, respectively, a complete system or a basis in G.

Let us note that if $\{g_n\}$ is a basis in G, then the spaces of coefficients of expansions both with respect to the functions $\{g_n\}$ and with respect to the functions $\{f_n = T g_n\}$ coincide. In particular, if $G = L_2$, then a basis $\{g_n\}$ in L_2 also generates in L_2 a basis $\{f_n = T g_n\}$ with the same space of expansion coefficients.

Let us also note that from the very definition of the set G it follows that a system of functions $\{f_n\}$ complete in G is complete in L_2.

§ 2. $G = L_2$. A system of functions $\{g_n\} \subset L_2$ is called **minimal** if none of them belongs to the closed linear span of the others. It is easy to show that the system $\{g_n\}$ is minimal if and only if from

$$\lim_{n\to\infty} \left| c_1^{(n)} g_1 + c_2^{(n)} g_2 + \cdots + c_n^{(n)} g_n \right| = 0$$

it follows that

$$\lim_{n\to\infty} c_i^{(n)} = 0, \qquad i = 1, 2, \ldots .$$

S. S. Levin proved ([1]) that if the system $\{g_n\}$ is minimal, then there exists a sequence of functions $\{h_n\}$ forming with $\{g_n\}$ a biorthogonal system, i.e.

$$(g_n, h_k) = \int_a^b g_{nh} k \, dx = \delta_{nk}, \qquad \delta_{nk} = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

Let $\{g_n\}$ be a complete minimal system of functions, and let the sequence of functions $\{R_n\}$, $R_n \in L_2$, $n = 1, 2, ...$, be such that the double series converges ...

series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (R_i, R_k)(h_i, h_k), \tag{1}$$

$(g_n, h_k) = \delta_{nk}$; then the system of functions $\{f_n = g_n + \lambda R_n\}$ is complete in L_2 for all regular values λ of the integral equation

$$f(x) = g(x) + \lambda \int_a^b K(x, s) \, g(s) \, ds \tag{2}$$

with kernel

$$K(x, s) = \sum_{i=1}^{\infty} R_i(x) h_i(s). \tag{3}$$

Indeed, in this case

$$\int_a^b \int_a^b [K(x, s)]^2 \, dx \, ds = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (R_i, R_k)(h_i, h_k) < \infty,$$

and therefore the integral equation (2) defines a linear operator $T_\lambda = J + \lambda K$, mapping L_2 into itself. It remains to note that $f_n = T_\lambda g_n$, and, if λ is a regular value of T_λ, to apply Theorem 1.

In particular, the system of functions $\{f_n = g_n + R_n\}$ is complete in L_2 if

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (R_i, R_k)(h_i, h_k) < 1.$$

A system of functions $\{f_n\}$, $n = 1, 2, ...$, is called linearly independent in L_2 (l.i. in L_2) if from $\sum_{n=1}^{\infty} c_n f_n = 0$ it always follows that $c_n = 0$, $n = 1, 2, ...$ Every minimal system is l.i. in L_2; the converse, generally speaking, is false.

Lemma 1. If $\{g_n\}$ is a basis (or a complete minimal system) in L_2, and the functions $\{f_n = g_n + \lambda R_n\}$ are l.i. in L_2 (respectively, also form a minimal system) and are such that the double series (1) converges, then λ is a regular value of the integral equation (2) with kernel (8).

Otherwise the integral equation

$$(J + \lambda K)g = 0 \tag{4}$$

has in L_2 a nontrivial solution g, $\|g\| \neq 0$. Representing g in the form

$$g = \sum_{n=1}^{\infty} c_n g_n$$

(respectively,

$$g = \lim_{n\to\infty} \left[c_1^{(n)} g_1 + c_2^{(n)} g_2 + \cdots + c_n^{(n)} g_n\right]$$

) and substituting this expansion into (4), we find that the system $\{f_n = g_n + \lambda R_n\}$ is not l.i. (respectively, minimal) in L_2, which contradicts what was assumed above.

From Lemma 1 it follows:

Theorem 2. Let the system of functions $\{f_n\}$ be l.i. in L_2 (or minimal), and let $\{g_n\}$, $n = 1, 2, ...$, be a basis (respectively, a complete minimal system) in L_2; if, moreover, the double series

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} (R_i, R_k)(h_i, h_k),$$

converges, where $R_n = f_n - g_n$, $(g_n, h_k) = \delta_{nk}$, then the system $\{f_n\}$ is a basis (respectively, complete) in L_2. The spaces of coefficients of expansions with respect to the systems $\{g_n\}$ and $\{f_n\}$ coincide.

The systems of functions $\{g_n\}$ and $\{f_n\}$ are called **quadratically close** if the series

$$\sum_{n=1}^{\infty} \|R_n\|^2, \qquad R_n = f_n - g_n$$

converges $((1), p.51)$, and B**-close** if

$$\sum_{i,k=1}^{\infty} |(R_i; R_k)| < \infty.$$

From what has been set forth, the following theorems on complete systems, obtained earlier by another method by N. K. Bari, follow as consequences.

1°. If $\{g_n\}$ is a complete orthonormal system, and the system $\{f_n\}$ is such that

$$\sum_{n=1}^{\infty} \|R_n\|^2 < 1, \qquad R_n = f_n - g_n,$$

then $\{f_n\}$ is a basis in L_2 with a space of expansion coefficients coinciding with l_2 $((2), p.68)$.

2°. If $\{g_n\}$ is a complete orthonormal system, and the system of functions $\{f_n\}$, quadratically close to $\{g_n\}$, is l.i. in L_2, then $\{f_n\}$ is a basis in L_2 with a space of expansion coefficients coinciding with l_2 $((2), p.72)$*.

We shall call the system $\{g_n\}$ **strictly minimal** if there exists a constant $\delta > 0$ such that the distance from any g_i to the closed linear span of the remaining functions $\{g_n\}$, $n \neq i$, is greater than or equal to δ for all $i = 1, 2, \ldots$. It is said that $\{g_n\}, \{h_n\}$ form a **regular biorthogonal system** if $\{g_n\}$ is a complete system and, in addition, the conditions

$$\sup_n \|g_n\| < \infty, \qquad \sup_n \|h_n\| < \infty$$

are simultaneously satisfied. A biorthogonal system $\{g_n\}, \{h_n\}$ is regular if and only if $\{g_n\}$ is a complete bounded (i.e.

$$\sup_n \|g_n\| < \infty$$

), strictly minimal system ([4]).

Using the results cited and Theorem 2, it is easy to obtain:

Theorem 3. If $\{g_n\}$ is a complete, bounded, strictly minimal system, and $\{f_n\}$ is a minimal system B-close to $\{g_n\}$, then $\{f_n\}$ is complete in L_2.

Corollary. Two B-close, strictly minimal systems are complete or incomplete simultaneously.

Theorem 4. If $\{g_n\}$ is a complete, bounded, strictly minimal system and

$$h = \sup_n \|h_n\|,$$

where

$$(g_n, h_k) = \delta_{nk},$$

and the system of functions $\{f_n\}$ is such that

$$\sum_{i,k=1}^{\infty} |(R_i; R_k)| < \frac{1}{h^2}, \qquad R_n = f_n - g_n,$$

then $\{f_n\}$ is complete in L_2.

A basis is a complete strictly minimal system. Therefore, if $\{g_n\}$ is a basis in L_2 and

$$\sup_n \|g_n\| < \infty,$$

and the system of functions $\{f_n\}$ is l.i. in L_2 and B-close to $\{g_n\}$, then $\{f_n\}$ is also a basis in L_2, with the same space of expansion coefficients as that of $\{g_n\}$.

It is easy to prove that when the systems of functions $\{g_n\}$ and $\{h_n\}$ form a complete biorthogonal sequence, i.e. both systems $\{g_n\}$ and $\{h_n\}$ are complete in L_2, in all the cases considered above there exists a system $\{F_n\}$ which forms, together with $\{f_n\}$, also a complete biorthogonal system. Indeed, in these cases the operator T, $f_n = Tg_n$, has a unique inverse T^{-1}. It remains to note that

$$F_n = (T^{-1})^* h_n,$$

where $(T^{-1})^*$ is the operator adjoint to T^{-1}.

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* In a later work, result 2° was extended by N. K. Bari to the so-called Riesz bases ([3]); however, it was not possible to obtain these generalizations from our theorems.

§ 3. Let us now consider the case where $G \neq L_2$. Let $\{g_n\}$, $n = 1, 2, ...$, be a complete orthonormal system of functions. Put $G = A_g$, where the set $A_g \subset L_2$ is such that if $f \in A_g$ and

$$f = \sum_{k=1}^{\infty} c_k g_k,$$

then the series

$$\sum_{k=1}^{\infty} |c_k|$$

converges. It is obvious that A_g, everywhere dense in L_2, is a space of type (B) with norm, for $f \in A_g$,

$$\|f\|_{A_g} = \sum_{k=1}^{\infty} |c_k|,$$

and satisfies all the conditions imposed on the set G in § 1.

Lemma 2. Let a system of functions $\{R_n\} \subset A_g$, $n = 1, 2, ...$, be such that

$$\sup_n |(R_n, g_k)| = \alpha_k,$$

and the series

$$\sum_{k=1}^{\infty} \alpha_k$$

converges.

The integral equation

$$f(x) = g(x) + \lambda \int_a^b K(x, s) g(s)\, ds$$

with kernel

$$K(x, s) = \sum_{i=1}^{\infty} R_i(x) g_i(s)$$

defines on A_g a linear operator

$$T_\lambda = I + \lambda K,$$

which maps A_g into itself and has in A_g a unique inverse for all values of λ, except those which coincide with the zeros of the entire function

$$\Delta(\lambda) = \begin{vmatrix} 1 + \lambda a_{11} & \lambda a_{12} & \cdots \\ \lambda a_{21} & 1 + \lambda a_{22} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix},$$

where $a_{ik} = (g_i, R_k)$.

(It can be shown, relying on Koch' s criterion, that $\Delta(\lambda)$ is an absolutely convergent determinant[5].)

Without particular difficulty one proves the following modifications of Theorem 1.

Theorem 5. If $\{g_n\}$ is a complete orthonormal system in L_2, and the system $\{f_n\}$ is such that

$$\sup_n |(R_n, g_k)| = \alpha_k, \qquad R_n = f_n - g_n, \qquad \sum_{k=1}^{\infty} \alpha_k < 1,$$

then $\{f_n\}$ is a basis in A_g with the space of coefficients of expansions coinciding with l_1.

Theorem 6. Let $\{g_n\}$ be a complete orthonormal system in L_2. Then, in order that the system of functions $\{f_n\}$,

$$\sup_n |(R_n, g_k)| = \alpha_k, \qquad R_n = f_n - g_n, \qquad \sum \alpha_k < \infty,$$

be a basis in A_g with the space of coefficients of expansions coinciding with l_1, it is necessary and sufficient that from

$$\sum_{n=1}^{\infty} c_n g_n = 0, \qquad \sum_{k=1}^{\infty} |c_k| < \infty$$

it always follow that $c_n = 0$ for all n.

In conclusion, we note that a system of functions $\{f_n\}$ complete in A_g is complete in L_2.

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Note: Figure translations are in progress. See original paper for figures.