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Abstract

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ON THE DIFFERENTIABILITY OF SOLUTIONS OF VARIATIONAL PROBLEMS IN NONPARAMETRIC FORM

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From the results of Bernstein ([1]), Lichtenstein ([2]), Hopf ([3]), and Morrey ([4]) it follows that if a function $z(x, y)$, realizing the minimum of the general regular double integral

$$\iint\limits_G F(x, y, z, p, q)\, dx\, dy, \qquad p = z_x,\ q = z_y, \tag{1}$$

satisfies the Lipschitz condition, then it has the same differentiability and analyticity properties as the integrand. On the other hand, under very general assumptions the existence of an absolutely continuous solution ([5–7]) of regular problems was proved. In this connection the following problem arose: under what conditions does a continuous solution of regular problems satisfy the Lipschitz condition. In ([7]) it is shown that such conditions may be certain restrictions on the order of growth of the integrand and of its derivatives with respect to $\sqrt{p^2 + q^2}$.

In the present note a new method for solving this problem is presented, making it possible to substantially improve the results of ([7]). At the same time this gives new results concerning the existence of a solution of the first boundary-value problem for equations of the calculus of variations.

1. Let the function $F(x, y, z, p, q)$ be defined and continuous for all $(x, y) \in G$, where G is a bounded two-dimensional domain with uniformly regular boundary* and for all z, p, q satisfy the conditions:

1,1) $F_{pp}F_{qq} - F_{pq}^2 > 0, \quad F_{pp} > 0$ everywhere in the domain of definition;

1,2) for $\sqrt{p^2 + q^2} = R \geq \Delta > 0$ there is a decomposition:

$$F = f(x, y, z) \cdot F^{(1)}(p, q) + F^{(2)}(x, y, z, p, q),$$

where $f > 0$, $F^{(1)}$ and $F^{(2)}$ are four-times differentiable functions (or analytic), and moreover:

1,3) the function $F^{(1)}(p, q)$ is positively homogeneous in p, q of degree $\alpha \geq 2$, and

$$F_{pp}^{(1)} F_{qq}^{(1)} - F_{pq}^{(1)2} \geq L > 0$$

for $p^2 + q^2 = 1$, where L does not depend on p and q;

1,4)

$$\left| \frac{\partial^\beta F^{(2)}(x, y, z, p, q)}{\partial p^m \partial q^n \partial z^k \partial x^l \partial y^s} \right| \leq L_1 R^{\alpha-\gamma-m-n},$$

where $m+n+k+l+s = \beta$; $\beta = 0, 1, 2, 3, 4$; $(x, y) \in G$; $|z| \leq z_0$; $L_1 = L_1(z_0) > 0$; $\gamma > 0$; $\sqrt{p^2 + q^2} \geq \Delta$.

Conditions 1,1)–1,4) are satisfied, for example, for the integrands considered in ([8]).

* The notion of a uniformly regular boundary was introduced in ([7]).

Theorem 1. Let the function F satisfy conditions 1,1)–1,4), and let $z_0(x, y) \in A^2$ (A^2 is the class of functions continuous in the sense of Tonelli, with finite Dirichlet integral $D(z) = \iint_G (z_x^2 + z_y^2)\, dx\, dy$)—a function realizing the minimum of the integral (1) for prescribed values on the boundary of the domain G. Suppose that the solution $z_0(x, y)$ is unique in the small*. Then the function $z_0(x, y)$ will satisfy a Lipschitz condition in every domain G_1, $\overline{G}_1 \subset G$, and, consequently, if $F \in C^{(2)}$ (or is analytic), then $z_0(x, y) \in C^{(2)}$ (respectively, is analytic) in every domain G_1, $\overline{G}_1 \subset G$.

A brief account of the proof of this theorem is contained in the following paragraphs.

2. Without loss of generality one may assume that in the expansion 1,2) the function $f \equiv 1$. To see this, transform the integral (1), putting:

$$u(x, y) = \int_0^{z_0(x,y)} [f(x, y, t)]^{1/4}\, dt.$$

3. A continuous function $z(x, y)$, $(x, y) \in D$, where D is a two-dimensional domain, will be called R-generalized-semiconcave ($R > 0$) if, for every domain $G \subset D$ and for all a and b satisfying the inequality $a^2 + b^2 \geq R^2$, the function $z(x, y) - ax - by$ attains its greatest and least values on the boundary of the domain G.

Lemma 1. Let $z(x, y)$ be an R-generalized-semiconcave function, given in the disk $K: \ x^2 + y^2 \leq 1$ and satisfying on the boundary of this disk a Lipschitz condition with constant L. Then the function $z(x, y)$ satisfies a Lipschitz condition with constant L_1 in every disk $K^{(\tau)}$ of radius $\tau < 1$, concentric with the disk K. The constant L_1 depends only on L, τ, and R.

4. A sequence of functions $\{F_n(x, y, z, p, q)\}$ will be called **majorizing with respect to the integrand** F, satisfying conditions 1,1)–1,4) (by paragraph 2 we assume $f \equiv 1$), if it has the following properties:

4,1) the functions F_n are defined for $(x, y) \in \overline{G}$ and for $|z| \leq z_0$ (z_0 does not depend on n) with arbitrary p, q, and are three times continuously differentiable, the third derivatives satisfying a Lipschitz condition in each domain

$$T = G_1 \times I^{(z_0)} \times K^{(R_0)}$$

of the five-dimensional space (x, y, z, p, q); here $G_1 \Subset G$, $I^{(z_0)} = (-z_0, +z_0)$; $K^{(R_0)}$ is the disk $p^2 + q^2 \leq R_0^2$, R_0 arbitrary;

4,2) for every $n = 1, 2, ...$ there exist numbers R_n and $\delta \geq 1$ (δ does not depend on n), $R_n \to \infty$, $n \to \infty$, such that inside the disk $p^2 + q^2 \leq R_n^2$ the function $F_n \equiv F$, while outside the disk $p^2 + q^2 \leq (\delta R_n)^2$ the function F_n does not depend on the point of the space (x, y, z);

4,3) $F_n(x, y, z, p, q) \geq F(x, y, z, p, q)$ for all $(x, y) \in \overline{G}$, $|z| \leq z_0$ and all p, q, and for any $n = 1, 2, 3, ...$;

4,4) the functions F_n satisfy condition 1,1), and for them, when $p^2 + q^2 \geq \Delta_1$ (Δ_1 does not depend on n), an expansion of the form 1,2) is valid; moreover, if

$$F_n \equiv F_n^{(1)}(p, q) + F_n^{(2)}(x, y, z, p, q),$$

then the functions $F_n^{(1)}$ and $F_n^{(2)}$ satisfy conditions 1,3) and 1,4), respectively, with constants γ, Δ_1, L, and L_1, independent of n.

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* That is, for every point $P \in G$ there is a disk K centered at this point, $\overline{K} \subset G$, on which $z_0(x, y)$ is the unique solution of the problem of the absolute minimum of the integral (1) among functions taking the same values on the boundary of K and lying in a sufficiently small neighborhood of $z_0(x, y)$.

Lemma 2. For any function $F(x, y, z, p, q)$ satisfying conditions 1.1)–1.4) ($f \equiv 1$), there exists a majorizing sequence of integrands $\{F_n\}$ with an arbitrarily prescribed value of the constant z_0.

5. Let $K^{(\varepsilon)}$, $\overline{K}^{(\varepsilon)} \subset G$, be a disk of radius $\varepsilon > 0$, chosen so small that the solution of the variational problem from Theorem 1 is unique in this disk. Let

$$z_n(x,y) \equiv n^2 \int_0^{1/n} \int_0^{1/n} z_0(x+t_1, y+t_2)\, dt_1 dt_2, \quad (x,y) \in K^{(\varepsilon)}$$

and

$$N_n = \max_{(x,y)\in K^{(\varepsilon)}} \{z_{nx}^2(x,y) + z_{ny}^2(x,y)\}^{1/2}.$$

Suppose that $N_n \to \infty$, $n \to \infty$; construct for the function $F(x,y,z,p,q)$ (F from Theorem 1) a majorizing sequence of functions F_n, for which

$$R_n \geqslant N_n + 1, \qquad z_0 = \max_{(x,y)\in K^{(\varepsilon)}} |z_0(x,y)| + \chi,$$

where $\chi > 0$ is an arbitrary number.

Let $z_n^0(x,y) \in A^2$, $(x,y) \in K^{(\varepsilon)}$, be the solution of the auxiliary variational problem determined by the integrand F_n, the domain $K^{(\varepsilon)}$, and the boundary values $z_n(x,y)$. From (7) it follows that the solution $z_n^0(x,y)$ exists, and moreover $|z_n^0(x,y)| \leqslant z_0$, $(x,y) \in K^{(\varepsilon)}$.

The sequence $\{z_n^0(x,y)\}$ satisfies the compactness criterion of work (7). Choosing from this sequence a convergent subsequence and using the uniqueness of the solution $z_0(x,y)$ and the majorizing property of the functions F_n, we establish that the chosen subsequence converges uniformly to the function $z_0(x,y)$, $(x,y) \in K^{(\varepsilon)}$; using property 4.2) of the function F_n, one can show that the function $z_n^1(x,y)$ will be (δR_n)-generalized saddle-shaped; hence, from Lemma 1 we conclude that each function $z_n^0(x,y)$ satisfies the Lipschitz condition in the disk $K^{(\varepsilon_1)}$, $0 < \varepsilon_1 < \varepsilon$, concentric with the disk $K^{(\varepsilon)}$, with a constant L_n depending on n.

6. Using the results of works ($^{2-4}$), we establish that there exists a disk $K^{(\varepsilon_2)}$, $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$, concentric with the disk $K^{(\varepsilon)}$, in which the functions $z_n^0(x,y)$, $n = 1, 2, ...$, will be three times continuously differentiable and, consequently, will satisfy the corresponding Euler–Lagrange equations of elliptic type. But then the uniform boundedness of all L_n (item 5) reduces to uniform a priori estimates of the partial derivatives of the solutions $z_n^0(x,y)$ of these equations.

Lemma 3. Let $z(x,y) \in C^{(3)}$, $(x,y) \in K^{(\varepsilon_2)}$, be a solution of the Euler–Lagrange equation corresponding to the function F, satisfying conditions 1.2)–1.4) and 4.1) (with $f \equiv 1$). Then

$$\max_{(x,y)\in K^{(\varepsilon_3)}} \{z_x^2(x,y) + z_y^2(x,y)\} \leqslant \frac{M}{(\varepsilon_2^2 - \varepsilon_3^2)^2}, \tag{2}$$

where $K^{(\varepsilon_3)}$, $\varepsilon_3 < \varepsilon_2$, is a disk concentric with the disk $K^{(\varepsilon_2)}$, and $M > 0$ is a constant depending only on

$$\max_{(x,y)\in K^{(\varepsilon_2)}} |z(x,y)| = z^{(\varepsilon_2)}$$

and on the constants L, L_1, γ, Δ (items 1, 3, 4). M does not depend on the function F.

The lemma is proved by the method developed by S. N. Bernstein in works ($^{8-10}$).

7. From the uniform boundedness of the sequence $\{z_n^0(x,y)\}$ and from $z_n^0(x,y) \in C^{(3)}$, $(x,y) \in K^{(\varepsilon_2)}$, $\varepsilon_2 < \varepsilon_1 < \varepsilon$, according to Lemma 3 it follows that there exists a disk $K^{(\varepsilon_3)}$, $\varepsilon_3 < \varepsilon_2$, concentric with $K^{(\varepsilon)}$, in which the inequality

(2) holds for all $z_n^0(x,y)$ uniformly, and M depends on

$$z_0 = \max_{(x,y)\in K^{(\varepsilon)}} |z_0(x,y)| + \chi = z^{(\varepsilon_2)},\ \Delta_1,\ \gamma,\ L \text{ and } L_1$$

and does not depend on n and F_n.

Consequently, $z_0(x,y)$ satisfies the Lipschitz condition in the disk $K^{(\varepsilon_2)}$. Hence we obtain that it satisfies the Lipschitz condition in every closed domain $G_1 \subset G$. Applying to $z_0(x,y)$ the results of papers (1-4), we obtain the assertion of Theorem 1.

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Note: Figure translations are in progress. See original paper for figures.