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Abstract

Full Text

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On Steady Wind Currents and the Distribution of Density in the Deep Sea

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The treatment of the question of wind-driven marine currents proposed in works ([1,2]), based on taking into account the process of turbulent diffusion of masses in an inhomogeneous deep sea, imposed certain restrictions both on the character of the stratification of the waters and on the form of the wind field over the sea. Here both of these restrictions are removed. We shall assume only that the distribution of density ρ in steady currents differs little from some stable distribution characterized by the function $\rho_*(z)$, i.e., that

$$\rho = \rho_0[\rho_*(z) + \delta(x, y, z)], \tag{1}$$

where ρ_0 is a characteristic value of the density of seawater ($\rho_0 \sim 1$ g/cm^3); δ is a small unknown perturbation of the density. Let

$$\frac{d\rho_*}{dz} = b\Phi(z), \tag{2}$$

where $b = \max(d\rho_*/dz)$ and, consequently, $\Phi(z) \leq 1$.

We take the linearized equations determining the velocity projections $(u,\ v,\ w)$ and the density of fluid particles (ρ) in the form ([2]):

$$-2\omega_z\rho_0 v = \mu_x \Delta u + \frac{\partial}{\partial z}\left(\mu_z \frac{\partial u}{\partial z}\right) - \frac{\partial p}{\partial x},$$

$$2\omega_z\rho_0 u = \mu_x \Delta v + \frac{\partial}{\partial z}\left(\mu_z \frac{\partial v}{\partial z}\right) - \frac{\partial p}{\partial y},$$

$$0 = g\rho - \frac{\partial p}{\partial z}, \tag{3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$\rho_0^2 w \frac{d\rho_*}{dz} = \mu_x \Delta\rho + \varepsilon \frac{\partial}{\partial z}\left(\mu_z \frac{\partial \rho}{\partial z}\right).$$

In contrast to what was adopted earlier ([2]), the continuity equation is taken here in the form customary for an incompressible fluid.

Let, on the free surface of the sea at $z = \zeta$,

$$\mu_z \frac{\partial u}{\partial z} = -T_x, \qquad \mu_z \frac{\partial v}{\partial z} = -T_y, \qquad \frac{\partial \delta}{\partial z} = 0, \qquad w = 0, \qquad p = 0. \tag{4}$$

At great depth, as $z \to \infty$,

$$u,\ v,\ \delta \to 0. \tag{5}$$

Far from the bounded region of action of the wind, as $\sqrt{x^2 + y^2} \to \infty$,

$$u,\ v,\ \delta \to 0. \tag{6}$$

In this case all boundary conditions (4), except the last one, may, with a sufficient degree of accuracy, be transferred to the undisturbed sea surface.

Using the equation of hydrostatics, we eliminate ρ from system (3). From the same equation there follows the expression for the sea level

$$\zeta = \int_0^\infty \delta \, dz. \tag{7}$$

We put

$$x = L\bar{x}, \qquad y = L\bar{y}, \qquad z = h\bar{z}, \qquad \mu_z = \mu_0 \bar{\mu}, \tag{8}$$

where L and h are characteristic scales, respectively, of the wind field and of the thickness of the friction layer (for example, D according to Ekman), while $\bar{\mu} = 1$ at $z = 0$, and introduce the parameters

$$\tau = \frac{\mu_x}{\mu_0} \frac{h^2}{L^2}, \qquad \alpha^2 = \frac{\mu_0^2}{4\omega_z^2 \rho_0^2 h^4}, \qquad \beta^2 = \frac{gbh^2}{4\omega_z^2 L^2}, \tag{9}$$

as well as the operators

$$M = \frac{d}{d\bar{z}}\, \bar{\mu} \frac{d}{d\bar{z}} + \tau\Delta, \qquad N = \varepsilon \frac{d}{d\bar{z}}\, \bar{\mu} \frac{d}{d\bar{z}} + \tau\Delta. \tag{10}$$

It is not difficult to show, using system (3), that the auxiliary unknown function

$$\omega(\bar{x}, \bar{y}, \bar{z}) = \int_{\infty}^{\bar{z}} \delta(\bar{x}, \bar{y}, \bar{z})\, d\bar{z} \tag{11}$$

satisfies the differential equation

$$[1 + \alpha^2 M^2] \frac{d}{d\bar{z}} \frac{N(\partial\omega/\partial\bar{z})}{\Phi(\bar{z})} + \beta^2 M \Delta\omega = 0. \tag{12}$$

Let us represent the prescribed components of the tangential wind stress on the sea surface, T_x, T_y, by means of definite integrals

$$T_x = T_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{X}(k, l) \exp[-i(k\bar{x} + l\bar{y})]\, dk\, dl,$$

$$T_y = T_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{Y}(k, l) \exp[-i(k\bar{x} + l\bar{y})]\, dk\, dl, \tag{13}$$

where T_0 is a characteristic magnitude of the wind stress. Correspondingly, we shall represent the unknowns of the problem in the form of sums of integrals

$$\omega(\bar{x}, \bar{y}, \bar{z}) = \sum_m \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C_m(k, l) f_m(\bar{z}) \exp[-i(k\bar{x} + l\bar{y})]\, dk\, dl,$$

$$u(\bar{x}, \bar{y}, \bar{z}) = \sum_m \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_m(k, l, \bar{z}) \exp[-i(k\bar{x} + l\bar{y})]\, dk\, dl, \tag{14}$$

$$v(\bar{x}, \bar{y}, \bar{z}) = \sum_m \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V_m(k, l, \bar{z}) \exp[-i(k\bar{x} + l\bar{y})]\, dk\, dl.$$

Similar expressions can also be constructed for w and ζ. In this case, according to (13) and (12), $f_m(\bar{z})$ is one of the four solutions of the equation

$$[1 + \alpha^2 M_1^2] \frac{d}{dz} \frac{N_1(f'(\bar{z}))}{\Phi(z)} - \beta^2 \xi M_1(f(\bar{z})) = 0, \tag{15}$$

satisfying the condition of decay as $z \to \infty$. The operators $M_1,\ N_1$ are obtained from $M,\ N$ in (10) by replacing Δ by $-\xi = -(k^2 + l^2)$. The summation in (14) is extended to the values of the index $m = 1, 3, 5, 7$.

The expressions (14) for $u,\ v$ satisfy the equations of motion (3), if one sets

$$U_m = Ak\psi_{1m}(\bar{z}) + Bl\psi_{2m}(\bar{z}) + DlC_m f_m(\bar{z}),$$

$$V_m = Al\psi_{1m}(\bar{z}) - Bk\psi_{2m}(\bar{z}) - DkC_m f_m(\bar{z}), \tag{16}$$

where

$$\psi_{1m} = C_m \frac{d}{dz}\frac{N_1(f')}{\Phi}, \qquad \psi_{2m} = M_1(\psi_{1m}); \tag{17}$$

$$A = -\frac{i\mu_0 L}{\rho_0 bh^3\xi}, \qquad B = -\frac{i\mu_0^2 L}{2\omega_z \rho^2 bh^5 \xi}, \qquad D = \frac{igh}{2\omega_z L}. \tag{18}$$

To determine $f_m(\bar{z})$ from (15) we shall apply approximate methods. In seeking slowly varying solutions it is natural to neglect the results of applying the operator $\frac{d}{dz}\bar{\mu}\frac{d}{dz}$ (and, all the more so, $\varepsilon\frac{d}{dz}\bar{\mu}\frac{d}{dz}$, bearing in mind that $\varepsilon \sim 0.1$) in comparison with $\tau\xi$. Thus, equation (15) reduces to the equation

$$\frac{d}{dz}\left(\frac{1}{\Phi}\frac{df_1}{dz}\right) - \lambda f_1 = 0, \qquad \lambda = \frac{\beta^2\xi}{1+\alpha^2\tau^2\xi^2}. \tag{19}$$

A particular solution of this equation under the boundary conditions $f_1'(0) = a\Phi(0)$, $f_1(\bar{z}) \to 0$ ($\bar{z} \to \infty$), can be found by the method of successive approximations. Restricting ourselves to the first approximation (according to an approximate estimate, $\lambda \ll 1$ for $\xi \leqslant 10$), we have:

$$f_1(\bar{z}) = a\int_\infty^{\bar{z}} \Phi(z)\,dz = \frac{a}{bh}(\rho_* - \rho_{z=\infty}). \tag{20}$$

In seeking solutions of equation (15) that vary rapidly with increasing $\bar{z}$, one may take $\Phi'(z) \ll f''(z)$, and therefore take $\Phi(z) \sim \Phi(0)$ for the surface layer of the sea, assuming that the density jump layer (if it exists) is located no higher than the friction layer. One may also allow several additional special simplifications in (15). In the estimates we proceed from the following values of the basic parameters: $L = 100$ km, $\omega_z = 5\cdot 10^{-5}$ sec^{-1}, $b = 5\cdot 10^{-6}$ cm^{-1}, $\mu_0 = 10^2$ g/cm $\cdot$ sec, $\mu_x = 10^8$ g/cm $\cdot$ sec. Hence (for $h = D = 45$ m) $\alpha^2 \sim 2.5\cdot 10^{-3}$, $\beta^2 \sim 0.1$, $\tau \sim 0.2$.

Equation (15) is now reduced to the equation:

$$\left[1 + \alpha^2\left(\frac{d}{dz}\bar{\mu}\frac{d}{dz} - \tau\xi\right)^2\right]\frac{d^2}{dz^2}\left[\tau\xi - \varepsilon\mu\frac{d^2}{dz^2}\right]f = 0. \tag{21}$$

Assuming, for simplicity, $\bar{\mu} = 1$, we find from this

$$f_3 = \exp\left[\sqrt{\frac{\tau\xi}{\varepsilon}}\,\bar{z}\right], \qquad f_5 = \exp\left[-\sqrt{\frac{\xi\tau a + i}{a}}\,\bar{z}\right],$$

$$f_7 = \exp\left[-\sqrt{\frac{\xi\tau a - i}{a}}\,\bar{z}\right]. \tag{22}$$

From the boundary conditions (4), after some simplifications we further find the value of the constant a entering into (20), and then also C_m $(m = 1, 3, 5, 7)$. For deep-water currents ($f_3, |f_5|, |f_7| \to 0$ for large z), it is sufficient to give the expression

$$C_1(k,l)f_1(\bar{z}) = \frac{T_0 L(\rho_* - \rho_\infty)i}{2\omega_z^2 \mu_x h}\,\frac{(l - k\xi\tau a)\widetilde{X} - (k + l\xi\tau a)\widetilde{Y}}{\xi(1 + a^2\tau^2\xi^2)}. \tag{23}$$

Using (22), we find expressions for the projections of the velocity of the deep-water (gradient-convection) currents

$$u = \frac{gT_0(\rho_* - \rho_\infty)}{4\omega_z^2\mu_x}\times$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\left[(l - k\xi\tau a)\widetilde{X} - (k + l\xi\tau a)\widetilde{Y}\right]\,[a\xi\tau k + l]\,\exp[-i(k\bar{x} + l\bar{y})]\,dk\,dl}{\xi(1 + a^2\tau^2\xi^2)^2}, \tag{24}$$

$$v = -\frac{gT_0(\rho_* - \rho_\infty)}{4\omega_z^2\mu_x}\times$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\left[(l - k\xi\tau a)\widetilde{X} - (k + l\xi\tau a)\widetilde{Y}\right]\,[a\tau\xi l - k]\,\exp[-i(k\bar{x} + l\bar{y})]\,dk\,dl}{\xi(1 + a^2\tau^2\xi^2)^2}.$$

It can be shown that the velocity of the geostrophic currents differs from the expressions found in (23) by a quantity of order $\alpha\tau = \mu_x/2\rho_0\omega_{zL}^2$ (~ 0.01).

Calculation of the velocity of deep-water currents for the case when the wind stress is confined to the square region $|\bar{x}| \le 1,\ |\bar{y}| \le 1$, where $T_x = 0,\ T_y = T_0$, leads, in particular, to the following results. At the center of the wind field ($\bar{x} = 0,\ \bar{y} = 0$), below the friction layer, we have

$$u = 0, \qquad v = \frac{gT_0(\rho_\infty - \rho_*)}{8\omega_z^2 \mu_x}. \tag{25}$$

At the depth where $\rho_\infty - \rho_* = 10^{-3}$, we find $v \sim 1.5$ cm/sec. On the OX axis, in this case,

$$u = 0, \qquad v = \pm\pi + 2\,\mathrm{arctg}\,\frac{\bar{x}^2}{2} \quad (|\bar{x}| \lesseqgtr 1), \tag{26}$$

whereas on the OX axis

$$u = 0, \qquad v = \pi - 2\,\mathrm{arctg}\,\frac{\bar{y}^2}{2}. \tag{27}$$

Far from the region of action of the wind, along the axes OX and OY, the quantity v varies according to the laws $v \sim -4/\bar{x}^2$ and, respectively, $v \sim 4/\bar{y}^2$.

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References Cited

[1] P. S. Lineikin, *DAN*, **101**, No. 3 (1955).
[2] P. S. Lineikin, *Fundamental Problems of the Dynamical Theory of the Baroclinic Layer of the Sea*, 1957.

Note: Figure translations are in progress. See original paper for figures.