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G. Ts. Tumarkin

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Abstract

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MATHEMATICS

G. Ts. Tumarkin

On the Behavior Near the Boundary of the Derivatives of Certain Sequences of Analytic Functions Converging Uniformly Inside a Domain

(Presented by Academician M. A. Lavrent' ev on 11 XII 1956)

1. In our note (4) a theorem was proved stating that if there is given a sequence of functions $\{f_n(z)\}$, analytic in the disk $|z| < 1$, satisfying the conditions:

a)

$$\int_0^{2\pi} \ln^+ \left| f_n\left(re^{i\theta}\right)\right| d\theta \leq C, \quad 0 < r < 1, \quad n = 1, 2, \dots;$$

b) on a set E, $mE > 0$, the sequence $\{f_n(e^{i\theta})\}$ of angular boundary values of $f_n(z)$ converges,

then from the sequence $\{f_n(z)\}$ one can select a subsequence $\{f_{n_l}(z)\}$ that will converge uniformly in closed subdomains of the disk $|z| < 1$ whose boundaries are rectifiable curves containing subsets of E of measure arbitrarily close to the measure of E. Let us note that in the well-known theorem of Khinchin-Ostrowski it is asserted that, when conditions a) and b) are fulfilled, uniform convergence of $\{f_n(z)\}$ takes place in closed subdomains lying entirely inside $|z| < 1$.

In the present note we use the theorem just stated, as well as certain results obtained by N. N. Luzin (1), Marcinkiewicz and Zygmund (5), and also Spencer (6), to study the behavior near the boundary of the sequence of derivatives $\{f_n'(z)\}$ of a given sequence of analytic functions.

2. Marcinkiewicz and Zygmund (5), generalizing results obtained by N. N. Luzin (1), showed that if $f(z)$ is an analytic function having angular boundary values on a set E, $mE > 0$, then for almost all points the integral

$$S(f,\theta) = \iint_{\Omega_\theta} |f'(z)|^2\, d\omega$$

is finite ($d\omega$ is the area element), which gives the magnitude of the area of the piece of the Riemann surface of the function $f(z)$ corresponding to the domain Ω_θ, bounded by a simple closed curve Γ_θ having with $|z| = 1$ only one common point $e^{i\theta}$; here it is assumed that Γ_θ near $z = e^{i\theta}$ lies between two chords of the unit circle drawn from $e^{i\theta}$ (Γ_θ is a non-tangential path to $|z| = 1$).

By α, $0 < \alpha < \pi$, in what follows we shall denote the angle between the tangents to $|z| = r$ drawn from the point $e^{i\theta}$ to $|z| = r$. For Ω_θ everywhere below we take the domain bounded by the just indicated tangents and the larger arc of the circle $|z| = r$. To emphasize the dependence of Ω_θ and $S(f,\theta)$ on α, we shall write $\Omega_{\theta,\alpha}$ and $S_\alpha(f,\theta)$.

Consider a sequence $\{f_n(z)\}$ of functions analytic in $|z| < 1$, each of which has angular boundary values $f_n(e^{i\theta})$ on a set E, and let $f(z)$ be a function analytic in $|z| < 1$ that also has angular boundary values on E. Then, for almost all points $e^{i\theta} \in E$, the expression $S(f - f_n, \theta)$ will be finite for all n, and one may ask under what additional conditions imposed on $\{f_n(z)\}$ convergence in the domains Ω_θ in the area mean of the derivatives of the given functions will take place.

Remark 1. In the work of N. N. Luzin ([1]), the question was also posed of the existence, for every function analytic in the disk $|z| < 1$,

$$F(z) = \sum_{k=0}^{\infty} \alpha_k z^k,$$

for which

$$\sum_{k=0}^{\infty} |\alpha_k|^2 < \infty$$

($F(z) \in H_2$), of such a closed Jordan rectifiable curve L, situated in $|z| \leqslant 1$ and having with the circle $|z| = 1$ a set of common points of positive measure, that

$$\iint_{\Omega_L} |F'(z)|^2\, d\omega < \infty$$

(Ω_L is the domain bounded by L). In the work of Rudin ([7]) an example is constructed of a function $F(z)$, analytic in $|z| < 1$ and continuous in $|z| \leqslant 1$, for which almost every radius of the disk $|z| < 1$ corresponds, on the Riemann surface of this function, to a curve of infinite length. It is not hard to see that for such a function there is no curve with the properties indicated above, and that the question posed by N. N. Luzin must be answered negatively. This also shows that, under the rather general assumptions considered by us concerning the functions of the sequence $\{f_n(z)\}$, one cannot, instead of the domains Ω_θ in which convergence in the mean of the derivatives of the given functions is studied, take subdomains of the disk $|z| < 1$ adjoining $|z| = 1$ on a set of positive measure.

Remark 2. The simplest examples (for instance $\{z^n/n\}$) show that, if we have a sequence $\{f_n(z)\}$ of functions analytic in the closed disk $|z| \leqslant 1$, converging uniformly in $|z| \leqslant 1$, it may turn out that one cannot choose a subsequence $\{f'_{n_k}(z)\}$, formed from the derivatives of these functions, which would converge uniformly even in a single domain Ω_θ. We note, moreover, that, assuming in advance the analyticity of the functions $f_n(z)$ only inside $|z| < 1$, as well as their continuity and single-valuedness in $|z| \leqslant 1$, we cannot guarantee the existence of angular boundary values for the derivatives of our functions. Examples of such single-valued functions $F(z)$ are given in the work of Lohwater, Piranian, and Rudin (9)*. Similar examples of single-valued and continuous functions in the disk $|z| \leqslant 1$, whose derivatives do not have angular boundary values almost everywhere on $|z| = 1$, are, as is not difficult to verify, provided by functions giving a conformal mapping of the disk $|z| < 1$ onto Jordan domains constructed by V. N. Veniaminov (8), possessing the property that to the set of boundary points of these domains attainable by angles there corresponds, under the conformal mapping, a set of points of the circle $|z| = 1$ of measure zero. Stronger examples of domains with analogous properties were constructed by M. A. Lavrent' ev in the work (2), where, instead of points attainable by angles, points attainable by line segments are considered.

§ 3. **Theorem 1.** *Let the sequence $\{f_n(z)\}$ of functions analytic in $|z| < 1$ satisfy the conditions of Khinchin-Ostrovskii a) and b) of § 1. Then from $\{f_n(z)\}$ one can choose a subsequence $\{f_{n_k}(z)\}$ such that:*

* The function $F(z)$ constructed by them is such that $F'(x)$ does not have even radial boundary values almost everywhere.

1. There is uniform convergence in each domain $\Omega_{\theta,\alpha}$ with vertex at almost all points $e^{i\theta} \in E$ and with any angle α, $0 < \alpha < \pi$, at the vertex.

2. In each of the indicated domains $\Omega_{\theta,\alpha}$

$$\lim_{n_k \to \infty} \iint_{\Omega_{\theta,\alpha}} \left| f'(z) - f'_{n_k}(z) \right|^2 d\omega = 0,$$

where

$$f(z) = \lim_{n \to \infty} f_n(z).$$

Proof. Assertion 1 of the theorem follows immediately from the assertion mentioned in item 1 of our theorem. We proceed to the proof of the second part of the theorem. For this we shall use the inequality proved by N. N. Luzin ([1])

$$\int_0^{2\pi} S_\alpha(F,\theta)\,d\theta \le A_\alpha \int_0^{2\pi} \left|F(e^{i\theta})\right|^2\,d\theta, \tag{1}$$

which holds for any function $F(z)$ belonging to the class H_2 in $|z|<1$, where the constant A_α depends only on α (see also ([5])).

Suppose first that the given sequence $\{f_n(z)\}$ converges uniformly in $|z|\le 1$ to the function $f(z)$. Applying inequality (1) to the function $f(z)-f_n(z)$, we obtain that for any α, $0<\alpha<\pi$,

$$\lim_{n\to\infty}\int_0^{2\pi} S_\alpha(f-f_n,\theta)\,d\theta = 0.$$

Hence it follows at once that one can select from $\{f_n(z)\}$ a subsequence $\{f_{n_k}(z)\}$ for which $\{S_\alpha(f-f_{n_k},\theta)\}$ will converge to zero for almost all θ and all α, $0<\alpha<\pi$.

We pass to the general case. Using our theorem on the possibility of extending to the boundary of the domain of uniform convergence for a subsequence of functions chosen from the given sequence $\{f_n(z)\}$ (see item 1), we obtain that there exists a subsequence $\{f_{n_i}(z)\}$ such that, for any $\varepsilon>0$, there will be found a subdomain $\overline{D}$ of the unit disk, bounded by a rectifiable Jordan curve Γ, with $m[\Gamma\cap E] > mE-\varepsilon$, in which $\{f_{n_i}(z)\}$ will converge uniformly. Let $z=\varphi(w)$ give a conformal mapping of the disk $|w|<1$ onto the domain D. Obviously, the sequence $\{f_{n_i}[\varphi(w)]\}$ will then converge uniformly in the closed disk $|w|\le 1$. By what was proved, one can select from this sequence a subsequence $\{f_{n_k}[\varphi(w)]\}$ such that, for almost all θ and for any α,

$$\lim_{n_k\to\infty}\iint_{\Omega_{\theta,\alpha}} \left|f'[\varphi(w)] - f'_{n_k}[\varphi(w)]\right|^2\,d\omega = 0.$$

Returning again to the domain D, and observing that in this case

$$\iint_{\Omega_{\theta,\alpha}} \left|f'[\varphi(w)] - f'_{n_k}[\varphi(w)]\right|^2\,d\omega = \iint_{\widetilde{\Omega}_{\theta,\alpha}} \left|f'(z) - f'_{n_k}(z)\right|^2\,d\omega,$$

where $\widetilde{\Omega}_{\theta,\alpha}$ is the image of $\Omega_{\theta,\alpha}$ under our mapping, and taking into account that under a conformal mapping of the disk onto a domain bounded by a rectifiable curve, sets of measure zero on the boundaries pass into sets of measure zero and conformality of angles holds at almost all boundary points, we readily verify that the subsequence $\{f_{n_k}(z)\}$ has the properties required in assertion 2 of the theorem.

Remark 1. Assertion 1 of Theorem 1 for a sequence of uniformly bounded functions was proved earlier by Zygmund ([10]).

Remark 2. One can construct examples showing that the assertions of Theorem 1 are valid only for subsequences of functions chosen from the given sequence, and need not hold for the whole sequence. In exactly the same way, examples show that the assertion that convergence takes place in the domains $\Omega_{\theta,\alpha}$ with vertices not at all points of E, but only almost everywhere on E, is exact.

Item 4. It is not difficult to see that assertion 2 of Theorem 1 holds whenever, for the sequence $\{f_n(z)\}$ of functions analytic in $|z| < 1$, assertion 1 of the same theorem is satisfied. Indeed, applying arguments similar to those used by Luzin and Privalov to prove the uniqueness theorem for analytic functions [(3)], we can select from our sequence a subsequence which will converge uniformly in closed subdomains of the circle $|z| < 1$ bounded by rectifiable curves and containing subsets of the set E of measure arbitrarily close to the measure of E (see [(4)]). In the proof of assertion 2 only this fact was used.

It can be shown that the converse proposition is also true: from the validity, for the sequence $\{f_n(z)\}$, of assertion 2 there follows the validity of assertion 1*. In proving this assertion one uses an estimate obtained by Spencer [(6)] for the proof of the theorem that if, for a function $f(z)$ analytic in the circle $|z| < 1$, the area of the piece of the Riemann surface corresponding to the triangle with vertex at each point $e^{i\theta} \in E$ is finite, then $f(z)$ has angular limiting values almost everywhere on E. In this way the following theorem can be established.

Theorem 2. *Let a sequence $\{f_n(z)\}$ of functions analytic in $|z| < 1$ converge uniformly inside $|z| < 1$ to $f(z)$. Then, in order that from $\{f_n(z)\}$ one can select a subsequence of functions converging uniformly in closed subdomains $\overline{D}$ of the circle $|z| < 1$, bounded by Jordan rectifiable curves and containing subsets of a prescribed set E on $|z| = 1$ of measure arbitrarily close to the measure of E, it is necessary and sufficient that there exist such a subsequence $\{f_{n_k}(z)\}$ of our sequence that for almost all $e^{i\theta} \in E$*

$$\lim_{n_k \to \infty} S(f_{n_k}, \theta) = S(f, \theta) < \infty. \tag{2}$$

Remark. It is obvious that always $\lim_{n \to \infty} S(f_n, \theta) \geq S(f, \theta)$.

The results given above are easily carried over from the case of the circle to domains bounded by rectifiable curves, and also extend to sequences of meromorphic functions.

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* In doing so we assume in advance that inside the circle $|z| < 1$ there was uniform convergence of $\{f_n(z)\}$; when condition 2 of Theorem 1 is satisfied, for this it is sufficient to require the convergence of $\{f_n(z)\}$ at one point z_0 of the circle $|z| < 1$. We note that one may in advance require convergence in the domains $\Omega_{\theta,\alpha}$ for $e^{i\theta} \in E$ not for all α; it is enough that for each $e^{i\theta} \in E$ there exist at least one $\alpha(\theta)$.

Note: Figure translations are in progress. See original paper for figures.

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