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Abstract

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POTENTIAL-HARMONIC FUNCTIONS AND SOME OF THEIR APPLICATIONS

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The notion of a function that is potential-harmonic in a certain domain was given in the work of A. V. Bitsadze ([1]). In the same work A. V. Bitsadze gave an analogue of the Cauchy formula for potential-harmonic functions in three-dimensional space. In the present paper, on the basis of results of V. K. Ivanov ([2]), an analogue of the Cauchy formula is obtained for potential-harmonic functions in n-dimensional space. On the basis of this formula a theorem is obtained on the relation between the supporting function of the convex hull of the singularities of a harmonic function regular at infinity in n-dimensional space and the growth indicatrix of the function associated with it. This theorem makes it possible to give a method of summing series of spherical functions. The case $n = 3$ was considered in ([3]).

1°. A vector function $\mathbf{f}(x_1, \dots, x_n) = f_1(x_1, \dots, x_n)\mathbf{i}_1 + \dots + f_n(x_1, \dots, x_n)\mathbf{i}_n$ is called potential-harmonic in a domain T if in this domain the following conditions are satisfied: 1) $\operatorname{div}\mathbf{f} = 0$; 2) $\frac{\partial f_k}{\partial x_j} = \frac{\partial f_j}{\partial x_k}$ for $k = 1, \dots, n$; $j = 1, \dots, n$. The triple product $[\mathbf{abc}]$ of vectors in n-dimensional space is defined as follows:

$$[\mathbf{abc}] = -(\mathbf{bc})\mathbf{a} + (\mathbf{ca})\mathbf{b} - (\mathbf{ab})\mathbf{c}.$$

Theorem 1. *If $\mathbf{f}(x_1, \dots, x_n)$, $\mathbf{g}(x_1, \dots, x_n)$ are potential-harmonic functions in a domain containing $T + \sigma$, where σ is the piecewise-smooth boundary of the domain T, then*

$$\int_\sigma [\mathbf{fng}]\, d\sigma = 0,$$

where $\mathbf{n}$ is the unit vector of the exterior normal to σ. If ρ is the distance between the points $M_1(x_1, \dots, x_n)$ and $M(\xi_1, \dots, \xi_n)$, ω_n is the surface area of the unit sphere, then

$$\frac{1}{\omega_n}\int_\sigma \left[\operatorname{grad}\frac{1}{\rho^{n-2}(\xi_1, \dots, \xi_n, x_1, \dots, x_n)}\,\mathbf{n}(\xi_1, \dots, \xi_n)\mathbf{f}(\xi_1, \dots, \xi_n)\right] d\sigma =$$

$$= \begin{cases} \mathbf{f}(x_1, \dots, x_n), & M_1(x_1, \dots, x_n) \in T, \\ 0, & M_1(x_1, \dots, x_n) \in T', \end{cases} \tag{1}$$

where T' is the domain complementing $T + \sigma$ to the whole space.

2°. A vector $\mathbf{p}(p_1, \dots, p_n)$, where $p_1^2 + \dots + p_n^2 = 0$, is called **isotropic**. We represent it in the following way ([2]): $\mathbf{p} = \mathbf{p}' + i\mathbf{p}''$, where $\mathbf{p}'$ and $\mathbf{p}''$ are real vectors, $i = \sqrt{-1}$. We have $\mathbf{p}' \perp \mathbf{p}''$, $|\mathbf{p}'| = |\mathbf{p}''| = \rho$. We shall assume that the vector $\mathbf{p}'$ issues from the origin O; then it will be determined if its length ρ and direction are known, which can

characterized by means of $\varphi_1, \dots, \varphi_{n-1}$—the angles of the spherical coordinate system. We shall assume that the vector $\mathbf{p}''$ also issues from the origin of coordinates O. It must lie in the $(n-1)$-dimensional space $V_{n-1}^{(\varphi_1, \dots, \varphi_{n-1})}$, orthogonal to the vector $\mathbf{p}'(\rho, \varphi_1, \dots, \varphi_{n-1})$, and will be completely determined by the angles $\psi_1, \dots, \psi_{n-2}$ of the spherical coordinate system in $V_{n-1}^{(\varphi_1, \dots, \varphi_{n-1})}$. Thus, the vector $\mathbf{p}$ is uniquely determined by specifying the quantities $\rho, \varphi_1, \dots, \varphi_{n-1}, \psi_1, \dots, \psi_{n-2}$.

Let the function $u(x_1, \dots, x_n)$ be regular harmonic outside a certain domain D containing the origin of coordinates; let σ be a piecewise-smooth surface enclosing all the singularities of $u(x_1, \dots, x_n)$.

Consider the function $F(\mathbf{p})$, defined as follows:

$$\mathbf{p}F(\mathbf{p}) = \frac{1}{\omega_n} \int_\sigma [\operatorname{grad} u(x_1, \dots, x_n)\, \mathbf{n}(x_1, \dots, x_n)\, \mathbf{p} e^{(\mathbf{pr})}]\, d\sigma, \tag{2}$$

where $\mathbf{n}$ is the unit vector of the exterior normal to σ; $\mathbf{p} = p_1\mathbf{i}_1 + \dots + p_n\mathbf{i}_n$ is an isotropic vector; $\mathbf{r} = x_1\mathbf{i}_1 + \dots + x_n\mathbf{i}_n$.

The functions $u(x_1, \dots, x_n)$ and $F(\mathbf{p})$ will be called associated. The function $F(\mathbf{p})$ is a function of exponential type; we define its growth indicatrix as follows:

$$h(\varphi_1, \dots, \varphi_{n-1}, \psi_1, \dots, \psi_{n-2}) = \overline{\lim_{\rho \to \infty}} \frac{\ln |F(\mathbf{p})|}{\rho}, \quad \text{where } \rho = |\mathbf{p}'| = |\mathbf{p}''|.$$

The support function of the domain D is the function

$$K(\varphi_1, \dots, \varphi_{n-1}) =$$

$$= \max_{(x_1, \dots, x_n) \in D} \{x_1 \cos\varphi_1 + x_2 \sin\varphi_1 \cos\varphi_2 + \dots + x_n \sin\varphi_1 \cdots \sin\varphi_n\},$$

where $0 \le \varphi_j \le \pi$ for $j = 1, \dots, n-2$; $0 \le \varphi_{n-1} \le 2\pi$.

Theorem 2. *The support function of the convex hull of the singularities of a function $u(x_1, \dots, x_n)$, harmonic and regular at infinity, is related to the growth indicatrix of the associated function $F(\mathbf{p})$ by the relation:*

$$K(\varphi_1, \dots, \varphi_{n-1}) = \sup_{\psi_1,\dots,\psi_{n-2}} h(\varphi_1, \dots, \varphi_{n-1}, \psi_1, \dots, \psi_{n-2}).$$

Indeed, the function $u(x_1, \dots, x_{n-1})$ outside a hypersphere with center at the origin and radius R, lying wholly in the domain of its regularity, can be represented by an absolutely and uniformly convergent series of spherical functions $r^{-p}H(m_k, +; \xi)$; $r^{-p}H(m_k, -; \xi)$ (4)

$$u(x_1, \dots, x_n) = \tag{3}$$

$$= \frac{1}{(n-2)\omega_n} \frac{1}{r^{n-2}} \sum_{p=0}^{\infty} \left(\frac{R}{r}\right)^p \sum [c_{m_0,\dots,m_{n-2}} H(m_k, +; \xi) + \bar{c}_{m_0,\dots,m_{n-2}} H(m_k, -; \xi)],$$

where the inner sum is taken over all integers m_k such that

$$p = m_0 \geq m_1 \geq m_2 \geq \dots \geq m_{n-2} \geq 0.$$

The expansion (3) may be replaced by the expansion

$$u(x_1, \dots, x_n) = \frac{1}{(n-2)\omega_n} \sum_{p=0}^{\infty} \left(\sum_{i_1+\dots+i_{n-1}=p} b_{i_1,\dots,i_{n-1},0} \frac{\partial^p}{\partial^{i_1} x_1 \cdots \partial^{i_{n-1}} x_{n-1}} \frac{1}{r^{n-2}} + \right.$$

$$\left. + \sum_{i_1+\dots+i_{n-1}=p-1} b'_{i_1,\dots,i_{n-1},1} \frac{\partial^p}{\partial^{i_1} x_1 \cdots \partial^{i_{n-1}} x_{n-1}\, \partial x_n} \frac{1}{r^{n-2}} \right), \quad r^2 = x_1^2 + \dots + x_n^2, \tag{3'}$$

whose coefficients are uniquely determined by the expansion (3).

Then from (1), (2) we obtain the following representation for the function $F(\mathbf{p})$:

$$F(\mathbf{p}) = \sum_{p=0}^{\infty} (-1)^p \left(\sum_{i_1+\dots+i_{n-1}=p} b_{i_1,\dots,i_{n-1},0} \rho_1^{i_1} \cdots \rho_{n-1}^{i_{n-1}} + \right.$$

$$\left. + \sum_{i_1+\dots+i_{n-1}=p-1} b_{i_1,\dots,i_{n-1},1} \rho_1^{i_1} \cdots \rho_{n-1}^{i_{n-1}} p_n \right). \tag{4}$$

We shall prove that the relation

$$u(x_1, \dots, x_n) = \frac{1}{2^{n-2}\pi^{\frac{n-2}{2}}\Gamma\left(\frac{n-2}{2}\right)} \int_{V_{n-1}^{(\varphi_1,\dots,\varphi_{n-1})}} F(\mathbf{p}) e^{-(\mathbf{pr})} \frac{dV}{\rho}, \tag{5}$$

is valid, where $V_{n-1}^{(\varphi_1,\dots,\varphi_{n-1})}$ is the $(n-1)$-dimensional space orthogonal to the vector $\mathbf{p}'(\rho, \varphi_1, \dots, \varphi_{n-1})$; $\rho = |\mathbf{p}'| = |\mathbf{p}''|$, $\mathbf{r} = x_1\mathbf{i}_1 + \dots + x_n\mathbf{i}_n$. If the vector $\mathbf{p}'$ has components $\mathbf{p}' = \{0, 0, \dots, 0, \rho\}$, then the space $V_{n-1}^{(0,\dots,0)}$ coincides with the space $OX_1 \dots X_{n-1}$, and the vector $\mathbf{p}''$ has components $\mathbf{p}' = \{x'_1, \dots, x'_{n-1}, 0\}$, where $(x'_1)^2 + \dots + (x'_{n-1})^2 = \rho^2$. On the basis of (5) and (6),

$$\int_{V_{n-1}^{(0,\dots,0)}} e^{-(\mathbf{pr})} \frac{dV}{\rho} = \int_{-\infty}^{\infty} \dots \int \frac{1}{\rho} e^{-\rho x_n} e^{-i(x'_1 x_1 + \dots + x'_{n-1} x_{n-1})}\, dx'_1 \cdots dx'_{n-1} =$$

$$= \frac{2^{n-2}\pi^{\frac{n-2}{2}}\Gamma\left(\frac{n-2}{2}\right)}{r^{n-2}}, \qquad n \geqslant 3.$$

Performing a rotation, we obtain that in the general case

$$\int_{V_{n-1}^{(\varphi_1,\dots,\varphi_{n-1})}} e^{-(\mathbf{pr})} \frac{dV}{\rho} = \frac{2^{n-2}\pi^{\frac{n-2}{2}}\Gamma\left(\frac{n-2}{2}\right)}{r^{n-2}}, \qquad n \geqslant 3. \tag{6}$$

Using (3), (4), (6), we obtain (5).

From formulas (2) and (5), the theorem formulated follows by arguments analogous to those carried out in the proof of Pólya' s theorem from the theory of entire functions ([5]).

Corollary. Suppose that all the singularities of a harmonic function $u(x_1, \dots, x_n)$ regular at infinity lie in the space $x_n < a$ $(a > 0)$. Denote by H the distance from the plane $x_n = a$ to the convex hull of the set of these singularities. The equality

$$H = a - \sup_{\psi_1,\dots,\psi_{n-2}} h(0, \dots, 0, \psi_1, \dots, \psi_{n-2}),$$

holds; h is the growth indicatrix of the function $F(\mathbf{p})$ associated with $u(x_1, \dots, x_n)$.

3°. Making in formula (5) the transformation of reciprocal radius-vectors, we obtain:

$$f(x_1, \dots, x_n) = \frac{1}{r^{n-2}} u\left(\frac{x_1}{r^2}, \frac{x_2}{r^2}, \dots, \frac{x_n}{r^2}\right) =$$

$$= \sum_{p=0}^{\infty} r^p \sum_{p=m_0 \geq m_1 \geq \dots \geq m_{n-2} \geq 0} \left[a_{m_0,\dots,m_{n-2}} H(m_k, +; \xi) + \bar{a}_{m_0,\dots,m_{n-2}} H(m_k, -; \xi)\right] =$$

$$= \frac{1}{2^{n-2}\pi^{\frac{n-2}{2}}\Gamma\left(\frac{n-2}{2}\right)} \frac{1}{r^{n-2}} \int_{V_{n-1}^{(\varphi_1,\dots,\varphi_{n-1})}} F(\mathbf{p}) e^{-\left(\mathbf{p}\frac{\mathbf{r}}{r^2}\right)} \frac{dV}{\rho}, \qquad r^2 = x_1^2 + \dots + x_n^2. \tag{7}$$

The function $f(x_1, \dots, x_n)$ is regular, harmonic in a neighborhood of the origin; the series in (7) converges inside the hypersphere with center at the ori-

the coordinate origin O, passing through the singular point of the function $f(x_1, \dots, x_n)$ nearest to O.

We shall call a point $M(r, \varphi_1, \dots, \varphi_{n-1})$ a **point of summability of the series** in (7) **by the method** (J') if the integral in (7) converges at this point. The domain of summability T of the series in (7) by the method (J') can be constructed analogously to the way in which the interior domain of the Borel polygon is constructed in summation by the Borel method (B') for series of functions of one complex variable. Namely, let $\mathfrak{M}$ be the set of all singular points P of the function $f(x_1, \dots, x_n)$. Then the domain of summability T is the set of all points M lying on one and the same side (with the coordinate origin O) of each hyperplane Q_P passing through P perpendicular to the ray OP.

If the boundary of the domain T is σ, and T' is the domain complementing $T + \sigma$ to the whole space, then the following proposition holds: *the series (7) is not summable by the method* (J') *at any point of the domain* T'.

What has been stated follows from the fact that the domain T is obtained from the domain T^*, bounded by the surface with equation $r = K(\varphi_1, \dots, \varphi_{n-1})$, where $K(\varphi_1, \dots, \varphi_{n-1})$ is the support function of the convex hull of the singularities of the function $u(x_1, \dots, x_n)$, by applying to the latter the transformation of reciprocal radius-vectors. The domain T^*, however, may be regarded as the set-theoretic sum of hyperspheres constructed as follows: each singular point M of the function $u(x_1, \dots, x_n)$ is joined to the coordinate origin O, and a hypersphere with diameter OM is constructed.

On the basis of (2) and (3), for $n = 3$ one can transform (7) into the form

$$f(x, y, z) = \sum_{n=0}^{\infty} r^n \sum_{m=-n}^{n} a_{nm} P^{|m|}(\cos\theta) e^{im\varphi} =$$

$$= \frac{1}{2\pi\psi} \int_0^{\infty} e^{-t} \left\{ \int_0^{2\pi} \sum_{n=0}^{\infty} t^n r^n \sum_{m=0}^{n} \frac{(-1)^m}{(n-m)!} (\cos\theta + i\sin\theta\sin\psi)^{n-m} \times \right.$$

$$\times \left[a_{nm} e^{im(\pi/2+\varphi)} (i\cos\psi - i\sin\theta - \cos\theta\sin\psi)^m + \right.$$

$$\left. \left. + \bar{a}_{nm} e^{im(\pi/2+\varphi)} (i\cos\psi + i\sin\theta + \cos\theta\sin\psi)^m \right] d\psi \right\} dt; \qquad a_{n(-m)} = \bar{a}_{nm}. \tag{8}$$

4°. Starting from formula (1), it is easy to carry over to n-dimensional space all the main propositions developed in the work of A. V. Bitsadze (1), on the basis of which, as A. V. Bitsadze does, one can, for example, obtain the solution of the integral equation:

$$\int_{-\infty}^{\infty}\iint \frac{p(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta}{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2} = f(x,y,z).$$

The required function $p(x,y,z)$ is

$$p(x,y,z) = \int_{-\infty}^{\infty}\iint \frac{\Delta f}{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2}, \qquad \Delta f = \frac{\partial^2 f}{\partial\xi^2}+\frac{\partial^2 f}{\partial\eta^2}+\frac{\partial^2 f}{\partial\zeta^2}.$$

Here we assume that the functions $p(x,y,z)$ and $f(x,y,z)$ are such that the improper integrals exist.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.