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MATHEMATICS

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1957

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Abstract

Full Text

MATHEMATICS

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ON SEMI-REDUCIBLE RIEMANNIAN SPACES

(Presented by Academician P. S. Aleksandrov, 16 III 1957)

1°. We shall call a Riemannian space V_n **semi-reducible** if there exists a coordinate system in which its metric form has the form

$$ds^2 = ds_0^2 + \sigma ds_1^2 = g_{ij}(x^k) dx^i dx^j + \sigma(x^k) a_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta$$

$$(i, j, k = 1, \dots, q; \alpha, \beta, \gamma = q + 1, \dots, n), \quad (1)$$

where ds_0^2 and ds_1^2 are independent q -dimensional and $(n - q)$ -dimensional metrics, each depending on its own arguments, while the function σ depends only on the variables of ds_0^2 . In particular, for $\sigma = \text{const}$, semi-reducible V_n include all reducible Riemannian spaces. We shall call the metric ds_0^2 the **principal part** of ds^2 , and ds_1^2 its **additional part**.

Semi-reducible V_n encompass a broad class of Riemannian spaces having the most diverse geometric properties. Thus, all spaces of constant curvature and Kagan subprojective spaces are semi-reducible, since they admit a coordinate system in which*

$$ds^2 = dx^{02} + \sigma(x^0) ds_1^2(x^1, \dots, x^{n-1}), \quad (2)$$

where ds_1^2 is a metric of constant curvature K_1 . The V_n of constant curvature are singled out from (2) by the condition $\sigma(\ln \sigma)'' = -2K_1$. In the study of motions in Riemannian spaces it turned out that all the most mobile V_n , with isolated exceptions, are semi-reducible ^(1,2). Semi-reducible V_n also appear in connection with conformal ⁽³⁾ and projective ⁽⁴⁾ transformations in V_n . Let us also note that in the theory of relativity the so-called centrally symmetric space-time ⁽⁵⁾, whose metric is likewise semi-reducible, is of great importance.

It is natural to pose the question of identifying properties characterizing the entire class of semi-reducible spaces V_n . In the present note a geometric characteristic is indicated and a tensor criterion for semi-reducible V_n is introduced; the question of the uniqueness of the representation of the metric of a proper Riemannian space in the form (1) is also considered.

2°. From (1) it is clear that a semi-reducible V_n is foliated into mutually orthogonal q -dimensional and $(n - q)$ -dimensional surfaces carrying the metrics ds_0^2 and

σds_1^2 . Here the first surfaces are totally geodesic, while the second are similar to one another and consist, as is not difficult to verify, of umbilical points. Such a foliation of V_n turns out also

* Exceptional subprojective spaces are not included here; their metric $ds^2 = 2dx^1 dx^2 + \sigma(x^2)(e_3 dx^3{}^2 + \dots + e_n^2 dx^n{}^2)$

and sufficient for its semi-reducibility, namely, the following theorem holds.

Theorem 1. *In order that a Riemannian space V_n be semi-reducible, it is necessary and sufficient that it be foliated into two families of q -dimensional and $(n - q)$ -dimensional mutually orthogonal surfaces; moreover, the surfaces of one family are totally geodesic, while the surfaces of the other are umbilical and mutually similar.*

3°. If in the space (1) one takes a symmetric tensor A_{ab} with components $A_{ij} = A_{i\alpha} = 0$, $A_{\alpha\beta} = \sigma a_{\alpha\beta}$, then it is easy to see that the equations

$$A_{ab,c} = -\frac{1}{2}(u_{aA_{bc}} + u_{bA_{ac}}), \quad (3)$$

$$A_{ac}A_b^c = A_{ab} \quad (a, b, c = 1, \dots, n), \quad (4)$$

are satisfied, where the comma denotes covariant differentiation in V_n ; u_a is the gradient of the function $u = \ln \sigma$. At the same time, evidently, $A_{ab} \neq \lambda g_{ab}$.

Conversely, suppose that for some symmetric tensor $A_{ab} \neq \lambda g_{ab}$ and for a gradient u_a equations (3) and (4) are satisfied. It follows from (4) that the matrix $A = \|A_b^a\|$ has simple elementary divisors. This means that at each point the tensors A_{ab} and g_{ab} can be simultaneously reduced to the form $g_{ab} = \delta_{ab}$, $A_{ab} = \rho_a \delta_{ab}$, where ρ_a are the roots of the equation $|A_{ab} - \rho g_{ab}| = 0$; δ_{ab} are the Kronecker symbols. Using equations (3), one can show that the principal planes corresponding to different roots ρ_α are holonomic; and then, passing to the special coordinate system determined by these planes, it is no longer difficult to show that the space V_n is semi-reducible.

Theorem 2. *In order that a Riemannian space V_n be semi-reducible, it is necessary and sufficient that there exist a symmetric tensor A_{ab} , not proportional to the metric tensor, which together with some gradient u_a satisfies the equations:*

$$1) \ A_{ab,c} = -\frac{1}{2}(u_{aA_{bc}} + u_{bA_{ac}}); \quad 2) \ A_{ac}A_b^c = A_{ab}.$$

This theorem is a direct generalization of the known criterion for reducibility⁽⁶⁾, which in our case is obtained for $u_a = 0$, i.e. $\sigma = \text{const}$. We note that for proper Riemannian spaces V_n ($ds^2 > 0$) condition 2) of the theorem is superfluous, in view of the fact that in this case the matrix $\|A_{ab} - \rho g_{ab}\|$ always

has simple elementary divisors. Consequently, *in a proper Riemannian space every nontrivial (i.e. $\neq \lambda g_{ab}$) symmetric tensor A_{ab} satisfying equations (3) determines a semi-reducible decomposition of the metric ds^2 .*

4°. Suppose that a semi-reducible space (1) is given. The question arises whether there exists another foliation of V_n into surfaces, geometrically distinct from the given one, as indicated in Theorem 1; that is, whether the representation of the metric ds^2 in the form (1) is unique up to trivial transformations of the form $\tilde{x}^i = f^i(x^k)$, $\tilde{x}^\alpha = \varphi^\alpha(x^\gamma)$.

We shall show by examples that uniqueness of the representation of the metric in semi-reducible form may fail:

$$1) \quad ds^2 = (ds_0^2 + \sigma_1 ds_1^2) + \sigma_2 ds_2^2 = (ds_0^2 + \sigma_2 ds_2^2) + \sigma_1 ds_1^2,$$

where ds_0^2 , ds_1^2 , ds_2^2 are independent metrics; σ_1, σ_2 depend only on the coordinates of the metric ds_0^2 .

$$2) \quad ds^2 = ds_0^2 + \sigma ds_1^2,$$

where $ds_1^2 = d\tau_0^2 + \nu d\tau_1^2$; then

$$ds^2 = (ds_0^2 + \sigma d\tau_0^2) + \sigma \nu d\tau_1^2.$$

$$3) \quad ds^2 = ds_0^2 + \sigma ds_1^2,$$

where

$$ds_0^2 = d\tau_0^2(x^1, \dots, x^p) + \lambda(x^1, \dots, x^p) d\tau_1^2(x^{p+1}, \dots, x^q),$$

$$\sigma = \lambda(x^1, \dots, x^p) \nu(x^{p+1}, \dots, x^q);$$

then

$$ds^2 = d\tau_0^2 + \lambda(d\tau_1^2 + \nu ds_1^2).$$

In the first case, several semi-reducible decompositions of the metric ds^2 appeared because, for one principal part ds_0^2 , it contained several additional metrics ds_α^2 . In the second example the principal part ds_0^2 increased owing to the semi-reducible decomposition of the additional metric ds_1^2 . We shall say that in this case the new semi-reducible decomposition of the metric ds^2 is a continuation to the right of the old one. Finally, in the third example we are dealing with a continuation to the left of the metric ds^2 , when the additional part increased owing to the semi-reducibility of ds_0^2 .

If one restricts oneself to proper Riemannian spaces, then it turns out that, for inhomogeneous ds_0^2 , there can be no other, more complicated examples of violation of uniqueness of the semi-reducible decomposition of the metric ds^2 . More precisely, the following theorem holds:

Theorem 3. *If the metric of a proper Riemannian space has, in some coordinate system, the form*

$$ds^2 = ds_0^2 + \sigma_1(x^i)ds_1^2 + \dots + \sigma_p(x^i)ds_p^2, \quad (5)$$

where ds_0^2 is an inhomogeneous and irreducible metric, and the functions $\sigma_1, \dots, \sigma_p$ are not pairwise in a constant ratio, then the corresponding foliation of the space V_n is unique, i.e. every other representation of ds^2 in the form (5) is obtained from the given one by trivial transformations.*

Thus, different semi-reducible decompositions of ds^2 of the form (1) are obtained from (5) either by choosing the additional metric ds_α^2 and assigning all the rest to the principal part, or by continuation to the right, if some metric ds_α^2 is itself semi-reducible.

If in (1) ds_0^2 is semi-reducible, then the metric ds^2 must be continued as far as possible to the left, and if the latter principal part proves to be inhomogeneous, then Theorem 3 can be applied, after first separating from the principal part all the remaining additional metrics ds_α^2 , if any exist.

5°. Let us next consider the case of a semi-reducible proper Riemannian space V_n with one-dimensional principal part

$$ds^2 = dx^{0^2} + \lambda(x^0)ds_1^2(x^1, \dots, x^{n-1}). \quad (6)$$

In this case Theorem 3 is not true. For example, on the Euclidean plane one can choose, in an infinite number of ways, a polar coordinate system in which $ds^2 = d\rho^2 + \rho^2 d\varphi^2$.

Lemma (cf. (3)). *In order that V_n be semi-reducible with one-dimensional principal part, it is necessary and sufficient that there exist a solution $f \neq \text{const}$, φ of the equations ***

$$f_{,ab} = \varphi g_{ab}. \quad (7)$$

Here it is necessary that $\varphi = \varphi(f)$.

Different representations of the given metric ds^2 in the form (6) are determined by different solutions of equations (7). Therefore, from all possible representations of the metric under consideration in semi-reducible form, there can

* I.e. transformations of coordinates inside each metric ds_0^2, \dots, ds_p^2 , a renumbering of the metrics, or replacement of σ_α by $c\sigma_\alpha$ and of ds_α^2 by $\frac{1}{c}ds_\alpha^2$.

** For an indefinite metric, when $\varphi = 0$, one must also require non-isotropy of the gradient $f_{,\alpha}$.

one canonical form may be chosen, namely the one in which ds_0^2 is determined by all functionally independent solutions of equations (7). If there is more than one such solution, then ds_0^2 has constant curvature K , and the function σ is such that $ds^{*2} = ds_0^2 + \sigma dt^2$ has the same constant curvature K . Such spaces V_n , in accordance with (4), will be denoted by $V_0(K)$.

Theorem 4. *If a semireducible V_n (1) admits nontrivial solutions of equations (7), then either it is a space $V_0(K)$, or every representation (1) of it is obtained from (6) by a right-hand continuation.*

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Received
15 III 1957

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