Physics

1957

SovietRxiv

Abstract

Full Text

Physics

B. B. KADOMTSEV

THE INVARIANCE PRINCIPLE FOR A HOMOGENEOUS MEDIUM OF ARBITRARY GEOMETRICAL SHAPE

(Presented by Academician M. A. Leontovich on 27 IX 1956)

1. The invariance principle in the theory of light scattering was first formulated by V. A. Ambartsumian for a semi-infinite medium ([1]). Subsequently S. Chandrasekhar extended it to the case of a plane layer ([2]). In the present work the invariance principle is applied to a medium of arbitrary geometrical shape, which is of interest both for the theory of light scattering and for certain problems of neutron physics.

To avoid unnecessary cumbersomeness, we shall assume that the scattering occurs without change of frequency (as will be seen below, the generalization to the case of scattering with change of frequency presents no special difficulties). In addition, we shall suppose that the medium under consideration is homogeneous and bounded by a convex surface S. By virtue of the homogeneity of the medium, the transport equation contains no explicit dependence on $\mathbf{r}$, and consequently it is invariant with respect to an infinitesimal displacement of space. Therefore the result of the action of such a transformation can be taken wholly into account by changing the boundary conditions. This statement constitutes the content of the invariance principle in the general case.

Taking as the unit of length the mean free path of a quantum, we write the transport equation for the influence function ([3]) in the form

$$\{(\omega\nabla) + \rho(\mathbf{r})\}G(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0) =$$

$$= \lambda\rho(\mathbf{r}) \int p(\omega \cdot \omega')G(\omega', \mathbf{r}; \omega_0, \mathbf{r}_0)\, d\omega' + \delta(\omega - \omega_0)\delta(\mathbf{r} - \mathbf{r}_0). \tag{1}$$

Here $G(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0)$ is the influence function, representing the intensity of radiation at the point $\mathbf{r}$ in the direction ω, due to a point source located at the point $\mathbf{r}_0$ and emitting radiation in the direction ω_0; λ is the absorption coefficient; $p(\cos\theta)$ is the scattering indicatrix; $\rho(\mathbf{r})$ is the "density of matter," equal to 1 inside S and 0 outside S.

Let us make an infinitesimal displacement of the medium by an amount ε in the direction of the unit vector $\mathbf{e}$, leaving the points of the source and of observation fixed. In this case the influence function $G(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0)$ passes into a new function $G'(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0)$, equal, evidently, to the old one, but with shifted source and observation points:

$$G'(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0) = G(\omega, \mathbf{r} - \varepsilon\mathbf{e}; \omega_0, \omega_0 - \varepsilon\mathbf{e}).$$

Thus the variation is

$$\delta G = G' - G = -\varepsilon\mathbf{e}(\nabla + \nabla_0)G(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0). \tag{2}$$

On the other hand, such a displacement is equivalent to a certain variation of the density, when matter is removed from one side and added on the other (which, in turn, is tantamount to a change in the boundary conditions). Varying (1), we obtain

$$\begin{aligned}\{(\omega\nabla) + \rho(\mathbf{r})\}\, \delta G(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0) - \rho(\mathbf{r}) \cdot \lambda \int p(\omega\omega')\, \delta G(\omega', \mathbf{r}; \omega_0, \mathbf{r}_0)\, d\omega' \\ = -\delta\rho(\mathbf{r}) \left\{ G(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0) - \lambda \int p(\omega\omega') G(\omega', \mathbf{r}; \omega_0, \mathbf{r}_0)\, d\omega' \right\}.\end{aligned} \tag{3}$$

As we see, the variation δG satisfies the transport equation and therefore can be represented through the influence function:

$$\begin{aligned}\delta G(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0) = -\int G(\omega, \mathbf{r}; \omega', \mathbf{r}')\, \delta\rho(\mathbf{r}')\, \{G(\omega', \mathbf{r}'; \omega_0, \mathbf{r}_0) - \\ -\lambda \int p(\omega'\omega'') G(\omega'', \mathbf{r}'; \omega_0, \mathbf{r}_0)\, d\omega'' \right\} d\omega'\, dr',\end{aligned}$$

where dr' denotes an element of volume. Substituting here (2) and taking into account that the variation of the density is equal to $\delta\rho = -\varepsilon\mathbf{e}\nabla\rho$ and is nonzero only on the surface S, where it has a δ-singularity, we obtain:

$$\begin{aligned}-\varepsilon\mathbf{e}(\nabla + \nabla_0)G(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0) = \\ = -\varepsilon\mathbf{e} \oint_S \int G(\omega, \mathbf{r}; \omega', \mathbf{r}_s) G(\omega', \mathbf{r}_s; \omega_0, \mathbf{r}_0)\, d\omega'\, dS + \\ +\varepsilon\mathbf{e}\lambda \oint_S \int G(\omega, \mathbf{r}; \omega', \mathbf{r}_s) p(\omega'\omega'') G(\omega'', \mathbf{r}_s; \omega_0, \mathbf{r}_0)\, d\omega'\, d\omega''\, dS,\end{aligned}$$

where $\mathbf{r}_s$ is the running radius vector over which the integration is carried out. The first integral on the right-hand side of this equation vanishes by virtue of the boundary conditions to which the influence function is subject:

$$G(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0) = 0 \quad \text{for} \quad \mathbf{r} \in S,\ \omega \cdot \mathbf{n} < 0 \quad \text{or} \quad \mathbf{r}_0 \in S,\ \omega_0 \cdot \mathbf{n}_0 > 0,$$

where $\mathbf{n}$ is the outward normal to S at the point $\mathbf{r}$, and $\mathbf{n}_0$ at the point $\mathbf{r}_0$. Taking into account that the vector $\mathbf{e}$ is arbitrary, we finally obtain:

$$(\nabla + \nabla_0) G(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0) = \\ = -\lambda \oint_S \int G(\omega, \mathbf{r}; \omega', \mathbf{r}_s) p(\omega'\omega'') G(\omega'', \mathbf{r}_s; \omega_0, \mathbf{r}_0)\, d\omega'\, d\omega''\, dS. \tag{4}$$

Let us now perform an infinitesimal rotation of space through an angle ξ about some point $\mathbf{R}$. Under this, G passes into

$$G'(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0) =$$

$$= G(\omega - [\xi\omega],\ \mathbf{r} - [\xi(\mathbf{r} - \mathbf{R})];\ \omega_0 - [\xi\omega_0],\ \mathbf{r}_0 - [\xi(\mathbf{r}_0 - \mathbf{R})]).$$

On the other hand, the change in the influence function can be taken into account by the variation of the density $\delta\rho = -[\xi(\mathbf{r} - \mathbf{R})] \cdot \nabla\rho(\mathbf{r})$. Carrying out calculations analogous to those given above, it is not difficult to obtain:

$$\left\{ [(\mathbf{r} - \mathbf{R})\nabla] + [(\mathbf{r}_0 - \mathbf{R})\nabla_0] + \left[\omega \frac{\partial}{\partial \omega}\right] + \left[\omega_0 \frac{\partial}{\partial \omega_0}\right] \right\} G(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0) = \\ = -\lambda \oint_S \int G(\omega, \mathbf{r}; \omega', \mathbf{r}_s) p(\omega'\omega'') G(\omega'', \mathbf{r}_s; \omega_0, \mathbf{r}_0)\, d\omega'\, d\omega'' [(\mathbf{r}_s - \mathbf{R})\, dS]. \tag{5}$$

The six scalar equations corresponding to the two vector equations (4), (5) constitute the mathematical formulation of the invariance principle. For obtaining these equations, the homogeneity of the optical properties proves to be very essential; otherwise the variation $\delta\rho$ would be different from zero throughout the entire volume inside S, and the integrals would not reduce to surface integrals.

Let us apply the equations obtained to the problem of diffuse reflection of light from a homogeneous medium. The solution of this problem reduces to finding the influence function $G(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0)$ under the condition that the points $\mathbf{r}$ and $\mathbf{r}_0$ lie on the surface S. Denote by ∇' the gradient along the surface S, defined by the formula $\nabla' = \nabla - \mathbf{n}(\mathbf{n}\nabla)$, where $\mathbf{n}$ is the outward normal to S at the point $\mathbf{r}$. Taking into account the identity $\nabla = \nabla' - \dfrac{\mathbf{n}}{\omega\mathbf{n}}(\omega\nabla') + \dfrac{\mathbf{n}}{\omega\mathbf{n}}(\omega\nabla)$ and using equation (1) and its adjoint (3), it is not difficult to reduce (4), (5) to the form:

$$\left\{\nabla' - \frac{\mathbf{n}}{\omega\mathbf{n}}(\omega\nabla') - \frac{\mathbf{n}}{\omega\mathbf{n}} + \nabla_0' - \frac{\mathbf{n}_0}{\omega_0\mathbf{n}_0}(\omega_0\nabla_0') + \frac{\mathbf{n}_0}{\omega_0\mathbf{n}_0}\right\} G(\omega,\mathbf{r};\omega_0,\mathbf{r}_0) =$$

$$= -\frac{\mathbf{n}}{\omega\mathbf{n}}\lambda \int p(\omega\omega') G(\omega',\mathbf{r};\omega_0,\mathbf{r}_0)\, d\omega' +$$

$$+ \frac{\mathbf{n}_0}{\omega_0\mathbf{n}_0}\lambda \int G(\omega,\mathbf{r};\omega',\mathbf{r}_0) p(\omega'\omega_0)\, d\omega' - \qquad (6)$$

$$-\lambda \oint_S G(\omega,\mathbf{r};\omega',\mathbf{r}_s) p(\omega'\omega'') G(\omega'',\mathbf{r}_s;\omega_0,\mathbf{r}_0)\, d\omega'\, d\omega''\, dS;$$

$$\left\{\left[(\mathbf{r}-\mathbf{R})\left(\nabla' - \frac{\mathbf{n}}{\omega\mathbf{n}}(\omega\nabla') - \frac{\mathbf{n}}{\omega\mathbf{n}}\right)\right] + \right.$$

$$+ \left[(\mathbf{r}_0-\mathbf{R})\left(\nabla_0' - \frac{\mathbf{n}_0}{\omega_0\mathbf{n}_0}(\omega_0\nabla_0') + \frac{\mathbf{n}_0}{\omega_0\mathbf{n}_0}\right)\right] +$$

$$\left. + \left[\omega\frac{\partial}{\partial\omega}\right] + \left[\omega_0\frac{\partial}{\partial\omega_0}\right]\right\} G(\omega,\mathbf{r};\omega_0,\mathbf{r}_0) =$$

$$= -\frac{[(\mathbf{r}-\mathbf{R})\mathbf{n}]}{\omega\mathbf{n}}\lambda \int p(\omega\omega') G(\omega',\mathbf{r};\omega_0,\mathbf{r}_0)\, d\omega' + \qquad (7)$$

$$+ \frac{[(\mathbf{r}_0-\mathbf{R})\mathbf{n}_0]}{\omega_0\mathbf{n}_0}\lambda \int G(\omega,\mathbf{r};\omega',\mathbf{r}_0) p(\omega'\omega_0)\, d\omega' -$$

$$-\lambda \oint_S \{G(\omega,\mathbf{r};\omega',\mathbf{r}_s) p(\omega'\omega'') G(\omega'',\mathbf{r}_s;\omega_0,\mathbf{r}_0)\, d\omega'\, d\omega''[(\mathbf{r}_s-\mathbf{R})\, dS]\}\,.$$

This system of equations contains no dependence on interior points, since $\mathbf{r}$ and $\mathbf{r}_0$ lie on the surface and all derivatives are taken along the surface. Thus, in the present problem, as also in the one-dimensional case, the invariance principle leads to a closed system of equations. These equations are, generally speaking, only necessary conditions to which the influence function for a homogeneous medium must be subject. The question of the sufficiency of these equations for solving the problem, in other words, the question of the uniqueness of the solution, requires additional investigation.

In the practical application of the equations obtained, it is necessary first to separate out from G the δ-singularity caused by light quanta that have not undergone scattering. We do not carry out the separation of this singularity here, so as not to complicate the formulas.

2. If the scattering is isotropic, i.e. $p(\cos\theta) = \dfrac{1}{4\pi}$, then from (4)–(7) one can obtain simpler equations. Let us integrate, for example, equations (4), (5) with respect to ω_0 and express ∇ again through the gradient along the surface. Introducing the influence function G_0 with an isotropic source,

$$G_0(\omega,\mathbf{r};\mathbf{r}_0) = \frac{1}{4\pi}\int G(\omega,\mathbf{r};\omega_0,\mathbf{r}_0)\, d\omega_0,$$

we obtain for it the following equations:

$$\left\{\nabla' - \frac{\mathbf{n}}{\omega\mathbf{n}}(\omega\nabla') - \frac{\mathbf{n}}{\omega\mathbf{n}} + \nabla_0\right\} G_0(\omega, \mathbf{r}; \mathbf{r}_0) = \\ = -\frac{\mathbf{n}}{\omega\mathbf{n}}\frac{\lambda}{4\pi}\int G_0(\omega', \mathbf{r}; \mathbf{r}_0)\, d\omega' - \lambda \oint_S G_0(\omega, \mathbf{r}; \mathbf{r}_s) \int G_0(\omega', \mathbf{r}_s; \mathbf{r}_0)\, d\omega'\, dS; \tag{8}$$

$$\left\{\left[(\mathbf{r}-\mathbf{R})\left(\nabla - \frac{\mathbf{n}}{\omega n}(\omega\nabla') - \frac{\mathbf{n}}{(\omega\mathbf{n})}\right)\right] + [(\mathbf{r}_0 - \mathbf{R})\nabla_0] + \left[\omega\frac{\partial}{\partial\omega}\right]\right\} G_0(\omega, \mathbf{r}; \mathbf{r}_0) = \\ = -\frac{[(\mathbf{r}-\mathbf{R})\mathbf{n}]}{\omega n}\frac{\lambda}{4\pi}\int G_0(\omega', \mathbf{r}; \mathbf{r}_0)\, d\omega' - \lambda \oint_S G_0(\omega', \mathbf{r}; \mathbf{r}_s) \int G_0(\omega', \mathbf{r}_s; \mathbf{r}_0)\, d\omega'\, [(\mathbf{r}_s - \mathbf{R})\, dS]. \tag{9}$$

It is not difficult to see that, in the plane one-dimensional case, equation (8) coincides with equation (23) of paper ([4]) for the probability $p(\mu, x)$ of escape from the medium, at an angle $\arccos\mu$, of a quantum absorbed at depth x, the probability so defined differing from G_0 only by a constant factor. Therefore equations (8), (9) may be regarded as a generalization of V. V. Sobolev' s method to a medium of arbitrary shape. As we see, these equations are a simple consequence of the invariance principle.

If one integrates (4), (5) with respect to ω and with respect to ω_0, one can obtain still simpler equations:

$$(\nabla + \nabla_0) I(\mathbf{r}, \mathbf{r}_0) = -\lambda \oint_S I(\mathbf{r}, \mathbf{r}_s) I(\mathbf{r}_s, \mathbf{r}_0)\, dS, \tag{10}$$

$$\{[(\mathbf{r}-\mathbf{R})\nabla] + [(\mathbf{r}'_0 - \mathbf{R})\nabla_0]\} I(\mathbf{r}, \mathbf{r}_0) = -\lambda \oint_S I(\mathbf{r}, \mathbf{r}_s) I(\mathbf{r}_s, \mathbf{r}_0) [(\mathbf{r}_s - \mathbf{R})\, dS], \tag{11}$$

where $I(\mathbf{r}, \mathbf{r}_0)$ is defined by the formula

$$I(\mathbf{r}, \mathbf{r}_0) = \frac{1}{4\pi}\int G(\omega, \mathbf{r}; \omega_0, \mathbf{r}_0)\, d\omega\, d\omega_0$$

and represents the radiation density at the point $\mathbf{r}$, caused by the presence of a point isotropic source at the point $\mathbf{r}_0$.

Equations (10), (11) likewise constitute only necessary conditions to which the function $I(\mathbf{r}, \mathbf{r}_0)$ must be subject; therefore one cannot count on their making it possible to obtain the complete solution. Nevertheless, in some cases they may prove useful.

For example, in the problem of a plane isotropic source in a semi-infinite medium, equation (10), taking into account the symmetry condition $I(x, x_0) = I(x_0, x)$, can be transformed into the form:

$$I(x, x_0) = I_0(|x - x_0|) + \frac{\lambda}{2}\int_{|x-x_0|}^{x} I(x', 0) I(x' - x + x_0, 0)\, dx' +$$
$$+ \frac{\lambda}{2}\int_{|x-x_0|}^{x_0} I(x', 0) I(x' - x_0 + x, 0)\, dx', \qquad (12)$$

where $I(x, x_0)$ denotes the radiation density from a plane source; the distances x, x_0 are measured from the boundary of the medium, and I_0 is an arbitrary function. But from the form of (12) it follows that I_0 is the solution for a plane source in an infinite medium. If I_0 is regarded as known ([5],[6]), then (12) makes it possible to determine $I(x, x_0)$.

I express my gratitude to Prof. D. I. Blokhintsev for discussing the work.

Received
26 IX 1956

CITED LITERATURE

[1] V. A. Ambartsumyan et al., *Theoretical Astrophysics*, 1952. [2] S. Chandrasekhar, *Radiative Transfer*, IL, 1953. [3] B. B. Kadomtsev, DAN (in press). [4] V. V. Sobolev, *Astr. Zhurn.*, 28, 355 (1951). [5] W. Bothe, *Zs. f. Phys.*, 122, 648 (1944). [6] R. Marshak, H. Brooks, H. Hurwitz, *Nucleonics*, 4, 10 (1949).

Note: Figure translations are in progress. See original paper for figures.