ON IRREDUCIBLE LINEAR REPRESENTATIONS OF THE FULL LORENTZ GROUP

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Abstract

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ON IRREDUCIBLE LINEAR REPRESENTATIONS OF THE FULL LORENTZ GROUP

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In a note by the author ([1]), an exact formulation and solution were given of the problem of describing, up to equivalence, all completely irreducible representations of the proper Lorentz group. In the present note an analogous formulation and solution are given for the full Lorentz group; here, without any qualifications, the notation and results of note ([1]) are used.

A non-rigorous derivation of the formulas for representations of the full Lorentz group in infinitesimal form was given earlier in the paper ([2]) by I. M. Gelfand and A. M. Yaglom*; the formulas presented in the present note for representations of the full Lorentz group in integral form apparently appear here for the first time.

1. Formulation of the problem. Let $\mathfrak{G}$ denote the full Lorentz group, and $\mathfrak{G}_+$ the proper Lorentz group. Then $\mathfrak{G} = \mathfrak{G}_+ \cup s\mathfrak{G}$, where s is reflection with respect to the first three coordinate axes

$$s: \ x_1' = -x_1, \quad x_2' = -x_2, \quad x_3' = -x_3, \quad x_4' = x_4.$$

Here s commutes with all rotations of three-dimensional space and, moreover, satisfies the conditions: $s^2 = 1$; $sb(t)s = b(-t)$, where

$$b(t) = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \operatorname{ch} t & \operatorname{sh} t \\ 0 & 0 & \operatorname{sh} t & \operatorname{ch} t \end{Vmatrix}.$$

It follows that a representation $g \to T_g$ of the group $\mathfrak{G}$ is completely determined by its restriction $g \to T_g$ to the group $\mathfrak{G}_+$ and by the operator $S = T_s$, satisfying the conditions: $\alpha)$ $S^2 = 1$, $\beta)$ $ST_{b(t)}S = T_{b(-t)}$, $\gamma)$ $ST_g = T_{gS}$ for all rotations g of three-dimensional space.

Passing from representations of the group $\mathfrak{G}_+$ to representations of the group $\mathfrak{A}$, we can write these conditions in the form

$$S^2 = 1, \quad ST_\varepsilon S = T_{\varepsilon^{-1}} \quad \text{for all matrices } \varepsilon = \begin{Vmatrix} e^{-t} & 0 \\ 0 & e^t \end{Vmatrix}, \quad \operatorname{Im} t = 0 \qquad \text{(1a)}$$

and

$$ST_u = T_{uS} \quad \text{for all } u \in \mathfrak{U}. \qquad \text{(1b)}$$

Finally, these conditions can also be written in the form $ST_{gS} = T_{g^\wedge}$, where the notation is introduced: $g^\wedge = g^{*-1}$.

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* We note that for the case of unitary representations of the full Lorentz group the derivation ([2]) can be made rigorous, so that the non-rigorous character of this derivation essentially pertains to the case of nonunitary representations.

We shall call the group ring $\mathfrak{C}$ of the group $\mathfrak{G}$ the totality of all formal sums $c = x + sy$, where x, y belong to the group ring X of the group $\mathfrak{A}$; here the operations in $\mathfrak{C}$ are defined by the formulas

$$\lambda c = \lambda x + s(\lambda y), \qquad c_1 + c_2 = (x_1 + x_2) + s(y_1 + y_2),$$

$$c_1 c_2 = (x_1 x_2 + y_1^\wedge y_2) + s(y_1 x_2 + x_1^\wedge y_2)$$

for $c = x + sy,\ c_1 = x_1 + sy_1,\ c_2 = x_2 + sy_2$, where $x^\wedge(g) = x(g^\wedge)$. If a representation $g \to T_g$ of the group $\mathfrak{G}$ is given, then, putting $T_c = T_x + ST_y$ for $c = x + sy$, we obtain, as is easy to verify, a representation of the ring $\mathfrak{C}$.

A representation $g \to T_g$ of the group $\mathfrak{G}$ is called **completely irreducible** if the operators $T_c,\ c \in \mathfrak{C}$, form a completely irreducible set. The problem consists in describing, up to equivalence, all completely irreducible representations of the group $\mathfrak{G}$; here equivalence is understood in the sense of the definition given in (1).

2. **Construction of the representations.** We shall indicate a complete set of completely irreducible representations of the group $\mathfrak{G}$ that are inequivalent to one another. These representations are constructed as follows:

α) **Representations $D^+_{0,\rho},\ D^-_{0,\rho}$.** Let, in general, $S_{m,\rho}$ denote the completely irreducible representation $a \to T_a$ of the group $\mathfrak{A}$, defined by the numbers (m, ρ) (infinite-dimensional for $\rho^2 \neq -(|m|+2n)^2,\ n = 1, 2, ...$, and spinorial for $\rho^2 = -(|m|+2n)^2,\ n = 1, 2, ...$), and let $R_{m,\rho}$ be the space of this representation.

Consider the representation $S_{0,\rho}$ of the group $\mathfrak{A}$, and define in $R_{0,\rho}$ the operator S by one of the two formulas

$$Sf(u) = f(su), \qquad Sf(u) = -f(su), \tag{2}$$

where*

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is not hard to verify that each of these operators S satisfies conditions (1a), (1b) and, consequently, these operators together with the representation $S_{0,\rho}$ of the group $\mathfrak{A}$ define two representations of the group $\mathfrak{G}$, which we shall denote by $D^+_{0,\rho}$, $D^-_{0,\rho}$, respectively.

If $\{\xi^k_p\}$ is the canonical basis in $R_{0,\rho}$, then $S\xi^k_p = (-1)^k \xi^k_p$ in the case of the representation $D^+_{0,\rho}$, and $S\xi^k_p = (-1)^{k+1}\xi^k_p$ in the case of the representation $D^-_{0,\rho}$.

β) **Representations $D^+_{m,0}$, $D^-_{m,0}$, $m > 0$.** Consider the representations $S_{m,0}$ and $S_{-m,0}$, $m > 0$, of the group $\mathfrak{A}$. These representations are unitary and unitarily equivalent. Let W be an isometric operator taking $S_{m,0}$ into $S_{-m,0}$, and let A_s be the operator defined by the formula $A_{sf}(u) = \varkappa f(su)$ for $f \in R_{m,0}$, where $\varkappa = (-1)^{m/2}$ for even m and $\varkappa = (-1)^{(m+1)/2}$ for odd m. Then A_s isometrically maps $R_{m,0}$ onto $R_{-m,0}$. We now define the operator S in $R_{m,0}$ by one of the two formulas $S = A_{sW}$, $S = -A_{sW}$. Again one can verify that in either of these two cases the operator S, together with the representation $S_{m,0}$ of the group $\mathfrak{A}$, defines a representation of the group $\mathfrak{G}$. The two representations thus obtained will be denoted, respectively, by $D^+_{m,0}$, $D^-_{m,0}$.

If $\{\xi^k_p\}$ is the canonical basis in $R_{m,0}$, then $S\xi^k_p = (-1)^{[k]}\xi^k_p$ in the case of the representation $D^+_{m,0}$, and $S\xi^k_p = (-1)^{[k]+1}\xi^k_p$ in the case of the representation $D^-_{m,0}$, where $[k]$ denotes the integer part of the number k.

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* We denote by the same letter s the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and the reflection with respect to the first three axes; this, however, will not lead to any misunderstanding in what follows.

γ) Representations $D_{m,\rho}$, $m > 0$, $\rho \neq 0$. Consider the representations $S_{m,\rho}$, $S_{-m,\rho}$ of the group $\mathfrak{A}$. Let $\overline{R}_{m,\rho}$ be the unitary space which is the direct sum of the unitary spaces $R_{m,\rho}$, $R_{-m,\rho}$; define in $\overline{R}_{m,\rho}$ the operators T_a and S by setting

$$T_a\{f_1, f_2\} = \{T'_a f_1, T''_a f_2\}, \quad S\{f_1(u), f_2(u)\} = \varkappa\{f_1(su), (-1)^m f_2(su)\},$$

where T'_a, T''_a are the operators of the representations $S_{m,\rho}$, $S_{-m,\rho}$, and $\varkappa$ is defined in the same way as in case β).

It is easy to verify that the operators T_a and S define a representation of the group $\mathfrak{G}$; we denote this representation by $D_{m,\rho}$. If $\{\xi_p^k\}$, $\{\eta_p^k\}$ are the canonical bases of the representations $S_{m,\rho}$, $S_{-m,\rho}$, then the vectors $\overline{\xi}_p^k = \{\xi_p^k, 0\}$, $\overline{\eta}_p^k = \{0, \eta_p^k\}$ form a basis in $\overline{R}_{m,\rho}$, and the action of the operator S on these vectors is given by the formulas:

$$S\overline{\xi}_p^k = (-1)^{[k]}\overline{\eta}_p^k, \quad S\overline{\eta}_p^k = (-1)^{[k]}\overline{\xi}_p^k.$$

Theorem 1. The representations $D_{0,\rho}^+$, $D_{0,\rho}^-$, $D_{m,0}^+$, $D_{m,0}^-$, $D_{m,\rho}$ $(m > 0,\ \rho \neq 0)$, corresponding to all possible $m > 0$ and ρ, are completely irreducible and pairwise inequivalent.

Let us note that the operator S in each of these representations is unitary; consequently, these representations are unitary if and only if the corresponding $S_{m,\rho}$ are unitary. It is also evident that the representations $D_{0,\rho}^+$, $D_{0,\rho}^-$, $D_{m,0}^+$, $D_{-m,0}^-$, $D_{m,\rho}$ $(m > 0,\ \rho \neq 0)$ are finite-dimensional if and only if the corresponding representations $S_{m,\rho}$ are finite-dimensional.

3. Main result.

Theorem 2. Every completely irreducible representation of the full Lorentz group $\mathfrak{G}$ is equivalent to one of the representations

$$D_{0,\rho}^+,\ D_{0,\rho}^-,\ D_{m,0}^+,\ D_{m,0}^-,\ D_{m,\rho}\ (m > 0,\ \rho \neq 0).$$

4. Idea of the proof. Put $\mathfrak{C}_j^k = \{c = x + sy : x, y \in X_j^k\}$, $X_{j+}^k = \{x : x \in X_j^k,\ x^\wedge = x\}$, $X_{j-}^k = \{x : x \in X_j^k,\ x^\wedge = -x\}$; then $\mathfrak{C}_j^k$, X_{j+}^k are subrings of the rings $\mathfrak{C}$ and X_j^k, respectively.

Let $g \to T_g$ be a completely irreducible representation of the group $\mathfrak{G}$; let $\mathfrak{M}_j^k$ be some subspace different from (0) and corresponding to this representation. Then $\mathfrak{M}_j^k$ is invariant with respect to the operators T_c, $c \in \mathfrak{C}_j^k$. Put $A_c\xi = T_c\xi$ for $\xi \in \mathfrak{M}_j^k$. The correspondence $c \to A_c$ is a representation of the ring $\mathfrak{C}_j^k$.

I. If the representation $g \to T_g$ of the group $\mathfrak{G}$ is completely irreducible, then the corresponding representation $c \to A_c$ of the ring $\mathfrak{C}_j^k$ is also completely irreducible.

II. Two completely irreducible representations $g \to T'_g$, $g \to T''_g$ of the group $\mathfrak{G}$, for which $\mathfrak{M}_j'^k \neq (0)$, $\mathfrak{M}_j''^k \neq (0)$ for fixed j, k, are equivalent if and only if the corresponding representations $c \to A'_c$, $c \to A''_c$ of the ring $\mathfrak{C}$ are equivalent.

Now put $T_s = S$. Since $S^2 = 1$, we have $\mathfrak{M}_j^k = \mathfrak{P}_j^k \oplus \mathfrak{D}_j^k$, where $\mathfrak{P}_j^k = \{\xi : \xi \in \mathfrak{M}_j^k,\ S\xi = -\xi\}$, $\mathfrak{D}_j^k = \{\xi : \xi \in \mathfrak{M}_j^k,\ S\xi = \xi\}$; $\mathfrak{P}_j^k$, $\mathfrak{D}_j^k$ are invariant with respect to the operators T_x, $x \in X_{j+}^k$, and $T_x\mathfrak{P}_j^k \subset \mathfrak{D}_j^k$, $T_x\mathfrak{D}_j^k \subset \mathfrak{P}_j^k$ for $x \in X_{j-}^k$. Setting $\Lambda_x\xi = T_x\xi$ for $\xi \in \mathfrak{P}_j^k$, $x \in X_{j+}^k$, we obtain a representation $x \to \Lambda_x$ of the ring X_{j+}^k. Taking into account that $S = -1$ on $\mathfrak{P}_j^k$, we have:

III. If the original representation $g \to T_g$ is completely irreducible, then the representation $x \to \Lambda_x$ of the ring X_{j+}^k is also completely irreducible.

Since the ring X_j^k is commutative, we conclude from this that either $\mathfrak{P}_j^k = (0)$, or $\mathfrak{P}_j^k$ is one-dimensional. Similarly, either $\mathfrak{D}_j^k = (0)$, or $\mathfrak{D}_j^k$ is one-dimensional; consequently, if $\mathfrak{M}_j^k \neq (0)$, then $\mathfrak{M}_j^k$ is either one-dimensional or two-dimensional.

Let us consider these cases separately.

1) $\mathfrak{M}_j^k$ is one-dimensional. The representation $c \to A_c$ is then one-dimensional and therefore is defined by some multiplicative linear functional $\lambda(c)$ on $\mathfrak{C}_j^k$, and hence on X_j^k. To this latter functional there corresponds a certain representation $S_{m,\rho}$ of the group $\mathfrak{A}$. Since either $\mathfrak{P}_j^k = (0)$, or $\mathfrak{D}_j^k = (0)$, we have $\lambda(x) = 0$ on X_{j-}^k; consequently, $\lambda(x^*) = \overline{\lambda(x)}$. Hence we conclude that either $m = 0$, or $\rho = 0$; moreover, either $S = 1$ on $\mathfrak{M}_j^k$, or $S = -1$ on $\mathfrak{M}_j^k$. But then the functional $\lambda(c)$ for the given representation coincides with the functional $\lambda(c)$ for one of the representations $D_{0,\rho}^+$, $D_{0,\rho}^-$, $D_{m,0}^+$, $D_{m,0}^-$, so that, by II, the given representation is equivalent to one of these representations.

2) $\mathfrak{M}_j^k$ is two-dimensional; consequently, $\mathfrak{P}_j^k$, $\mathfrak{D}_j^k$ are one-dimensional. By virtue of the complete irreducibility of the representation $c \to A_c$ of the ring $\mathfrak{C}_j^k$, there exist a function $y_0 \in X_{j-}^k$ and vectors $\xi_1 \in \mathfrak{P}_j^k$, $\xi_2 \in \mathfrak{D}_j^k$ such that $A_{y_0}\xi_1 = \xi_2$, $A_{y_0}\xi_2 = \xi_1$. Moreover, from the commutativity of the ring X_j^k it follows easily that $A_x\xi_1 = \lambda'(x)\xi_1$, $A_x\xi_2 = \lambda'(x)\xi_2$ for $x \in X_{j+}^k$, and $A_x\xi_1 = \lambda'(xy_0)\xi_2$, $A_x\xi_2 = \lambda'(xy_0)\xi_1$ for $x \in X_j^k$, where $\lambda'(x)$ is a certain multiplicative linear functional in X_{j+}^k. At the same time $\lambda'(x_1y_0)\lambda'(x_2y_0) = \lambda'(x_1x_2)$, where $x_1, x_2 \in X_{j-}^k$.

Putting $\lambda(x) = \lambda(x' + x'') = \lambda'(x') + \lambda'(x''y_0)$ for $x' \in X_{j+}^k$, $x'' \in X_{j-}^k$, and taking into account the last relation for λ', we easily find that λ is a multiplicative linear functional in X_j^k, and consequently defines a certain representation $S_{m,\rho}$ of the group $\mathfrak{A}$. From the complete irreducibility of the representation $c \to A_c$ and the formulas

$$S\xi = -\xi \text{ on } \mathfrak{P}_j^k, \qquad S\xi = \xi \text{ on } \mathfrak{D}_j^k \tag{3}$$

we conclude that $m \neq 0$, $\rho \neq 0$, so that one may assume $m > 0$.

Moreover, from formulas (3) and the definition of the functional $\lambda(x)$ it follows that, for $c = (x_1 + y_1) + s(x_2 + y_2)$, $x_1, x_2 \in X_{j+}^k$, $y_1, y_2 \in X_j^k$, the matrix of the operator A_c in the basis ξ_1, ξ_2 has the form

$$\begin{Vmatrix} \lambda(x_1 - x_2) & \mu(y_1 + y_2) \\ \mu(y_1 - y_2) & \lambda(x_1 + x_2) \end{Vmatrix}.$$

But a direct computation shows that, with a suitable choice of basis, the matrix of the operator A_c for the representation $D_{m,\rho}$ has the same form. On the basis of Proposition II we therefore conclude that the given representation is equivalent to the representation $D_{m,\rho}$.

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References

[1] M. A. Naimark, DAN, 97, No. 6, 969 (1954). [2] I. M. Gelfand, A. M. Yaglom, ZhETF, 18, No. 8, 703 (1948).

Note: Figure translations are in progress. See original paper for figures.