# DIFFUSION OF A VORTEX LAYER AND HEAT TRANSFER

1957

SovietRxiv

## Abstract

## Full Text

## MATHEMATICAL PHYSICS

### S. A. REGIRER

# DIFFUSION OF A VORTEX LAYER AND HEAT TRANSFER

*(Presented by Academician S. L. Sobolev on 18 XII 1956)*

1. Let us consider the automodel problem of the diffusion of a vortex layer and heat transfer in the half-space $y > 0$, filled with a liquid whose viscosity, thermal conductivity, and heat capacity depend on temperature.

At the initial instant the liquid is at rest and has temperature $T = 0$. The plane $y = 0$ at the initial instant begins to move in the direction $z$ with constant velocity $U$, and its temperature instantaneously rises to $T = T_c$. In solving the problem we shall be especially interested in the question of the relation between the rates of decay of the initial disturbances—the thermal and the kinematic.

Assuming that the pressure throughout the half-space is a function only of time, $p = p(t)$, and taking the mass forces to be absent, we can write the equations of the problem in the form ($^1$):

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial y}\left(\nu \frac{\partial v}{\partial y}\right); \tag{1}$$

$$\frac{\partial (cT)}{\partial t} = \frac{\partial}{\partial y}\left(\frac{k}{\rho}\frac{\partial T}{\partial y}\right) + \frac{\nu}{J}\left(\frac{\partial v}{\partial y}\right)^2; \tag{2}$$

$$\nu = \nu_c \mu(T), \qquad k = k_c \chi(T), \qquad c = c_c \gamma(T), \tag{3}$$

where $\mu(T)$, $\chi(T)$, and $\gamma(T)$ are dimensionless functions.

From the system (1)–(3) it is necessary to find $v(y,t)$, $T(y,t)$, $\nu(y,t)$, $k(y,t)$, $c(y,t)$, satisfying the conditions:

$$\begin{array}{rlllll} t = 0: & v = 0, & T = 0, & \nu = \nu_n, & k = k_n, & c = c_n; \\ t > 0,\ y = 0: & v = U, & T = T_c, & \nu = \nu_c, & k = k_c, & c = c_c. \end{array} \tag{4}$$

The formulation of the problem leads to the conclusion that the required functions are represented in terms of the variables $t$, $y$ and the constants $T_c, U, \nu_c, \nu_n, Jk_c, Jk_n, Jc_c, Jc_n, \rho, \alpha_i$ (the $\alpha_i$ are constants entering into the equations of relation (3), one of these constants having the dimension of temperature and the others being dimensionless).

The only independent dimensionless combination of variables that can be formed from these parameters is a quantity of the type $r = y/\sqrt{2\nu_c t}$. Hence $v = U - u(r, A_i)$, $T = T_c\theta(r, A_i)$, $\nu = \nu_c\mu(r, A_i)$, $k = k_c\chi(r, A_i)$, $c = c_c\gamma(r, A_i)$, where $A_i$ are constant dimensionless parameters formed from the defining parameters of the problem. Denoting $P_c = \nu_c\rho c_c/k_c$ and $\Pi = \nu_c\rho U^2/Jk_{cT}c$, and transforming the system (1)–(4) to dimensionless ⋯

form, we obtain:

$$\frac{d}{dr}\left(\mu\frac{du}{dr}\right) + r\frac{du}{dr} = 0; \tag{5}$$

$$\frac{d}{dr}\left(\varkappa\frac{d\theta}{dr}\right) + P_c r\frac{d}{dr}(\gamma\theta) + \Pi\mu\left(\frac{du}{dr}\right)^2 = 0, \tag{6}$$

$$\mu = \mu(\theta), \qquad \varkappa = \varkappa(\theta), \qquad \gamma = \gamma(\theta); \tag{7}$$

$$u(0) = 1, \qquad u(\infty) = 0; \tag{8}$$

$$\theta(0) = 1, \qquad \theta(\infty) = 0. \tag{9}$$

Thus, the problem is reduced to a system of ordinary differential equations and can be solved numerically with the aid of a differential analyzer. In this sense the problem admits an exact solution. The system (5)–(9) is solvable in quadratures for the case of constant physical parameters of the liquid, $\mu = \varkappa = \gamma = 1$. Successive integration of equations (5) and (6), taking (8) and (9) into account, gives

$$u = 1 - \operatorname{erf}\left(\frac{r}{\sqrt{2}}\right), \tag{10}$$

$$\theta = 1 - \operatorname{erf}\left(r\sqrt{\frac{P_c}{\pi}}\right) + \frac{4\Pi}{\pi}\left\{J(P_c, \infty)\operatorname{erf}\left(r\sqrt{\frac{P_c}{2}}\right) - J(P_c, r)\right\}, \tag{11}$$

where

$$J(P_c, r) = \int_0^r \exp\left(-\frac{P_c\zeta^2}{2}\right)\left\{\int_0^\zeta \exp\left[\left(\frac{P_c}{2} - 1\right)\xi^2\right]d\xi\right\}d\zeta. \tag{12}$$

The integral $J(P_c, \infty)$ is evaluated by differentiating with respect to the parameter in closed form:

$$J(P_c, \infty) = \frac{1}{2} \frac{\ln\left[\sqrt{P_c(P_c-2)} + P_c - 1\right]}{\sqrt{P_c(P_c-2)}}. \tag{13}$$

Formula (13) is valid for any values of $P_c$; however, for $0 < P_c < 2$ it is more convenient to use it in the form

$$J(P_c, \infty) = \frac{1}{2} \frac{\operatorname{arc\,tg}\left[\sqrt{P_c(2-P_c)}/(P_c-1)\right]}{\sqrt{P_c(2-P_c)}}, \tag{14}$$

where $J(0,\infty) = \infty$, $J(1,\infty) = \pi/4$, $J(2,\infty) = 1/2$, and $J(P_c,\infty)$ is a monotonically decreasing function.

2. Let us now consider the behavior of the functions $u(r)$ and $\theta(r)$ in the region of large values of the time $t$ (for finite $y > 0$), which corresponds to very small values of $r$.

The character of the decay of the initial temperature and velocity disturbances is determined by the ratio $N = \left|\frac{1-\theta}{1-u}\right|$ for small $r$:

$$N = \lim_{r\to 0}\left|\frac{1-\theta}{1-u}\right| = \lim_{r\to 0}\left|\frac{\theta'}{u'}\right| = \lim_{t\to\infty}\left|\frac{\partial\theta/\partial t}{\partial u/\partial t}\right| = \sqrt{P_c}\left|\frac{4\Pi}{\pi}J(P_c,\infty) - 1\right|. \tag{15}$$

If $N > 1$, then the temperature disturbances caused by the combined action of the initial change in the temperature of the plane $y = 0$ and by energy dissipation decay faster than the velocity disturbances. If $N < 1$, the opposite phenomenon is observed.

The behavior of the function $N(P_c, \Pi)$ (15) is represented by the family of curves in Fig. 1 (solid curves). Analysis of formulas (13), (14), and (15) makes it possible to draw the following conclusions: a) in the absence of energy dissipation, the damping of the temperature disturbance is determined entirely by the Prandtl number $P_c$, with $N \geqslant 1$ for $P_c \geqslant 1$; b) in the presence of dissipation, flows of both types exist: $N > 1$ and $N < 1$.

The entire set of values of $P_c$ and $\Pi$ is divided into two classes by the curve $N(P_c, \Pi) = 1$ in the $(P_c, \Pi)$ plane (see Fig. 2).

It should be noted that in the problem under consideration both temperature disturbances are directed toward an increase in temperature. One may, however, pose the problem differently, by setting $\theta(0) = 0$, $\theta(\infty) = 1$, so that the cooling of the entire fluid caused by the initial decrease in the temperature of the plane $y = 0$

Fig. 1 and Fig. 2

Figure 1: Fig. 1 and Fig. 2

Fig. 1 Fig. 2

will be to some extent impeded by dissipative heating. For this case $N$ is determined by the formula

$$N = \lim_{r \to 0} \left| \frac{\theta}{1-u} \right| = \sqrt{P_c} \left| \frac{4\Pi}{\pi} J(P_c, \infty) + 1 \right|, \tag{16}$$

which, as was to be expected, gives larger values of $N$ than formula (15). For comparison, in Fig. 1 the curves $N(P_c, \Pi)$ according to (16) are plotted with dashed lines.

3. In the case of variable viscosity $\mu = \mu(\theta)$, generally speaking, one can obtain approximate estimates for the parameter $N$ by using the fact that $\mu$ is bounded above and below: $m \leqslant \mu \leqslant M$. In this case one of the bounds is obtained directly from conditions (9): if $\mu'(\theta) < 0$, then $\mu \geqslant \mu_{\theta=0} = m$; if $\mu'(\theta) > 0$, then $\mu \leqslant \mu_{\theta=0} = M$. The second bound for $\mu$ can be expressed approximately through the first and the numbers $P_c, \Pi$.

Taking $\mu(r)$ as a known function, we find the solution of system (5)–(9) for $x = \gamma = 1$. For the number $N$ we obtain the formula

$$N = \mu_{r=0} |A| \int_0^\infty \frac{1}{\mu} \exp\left( -\int_0^r \frac{\zeta}{\mu}\, d\zeta \right) dr, \tag{17}$$

where

$$A = \frac{\Pi \int_0^\infty \exp\left( -\frac{P_c r^2}{2} \right) \left[ \int_0^r \frac{1}{\mu} \exp\left( \frac{P_c \zeta^2}{2} - 2\int_0^\zeta \frac{\xi}{\mu}\, d\xi \right) d\zeta \right] dr}{\left[ \int_0^\infty \frac{1}{\mu} \exp\left( -\int_0^r \frac{\xi}{\mu}\, d\xi \right) dr \right]^2} - 1. \tag{18}$$

The temperature distribution is described by the equation

$$\theta = 1 + A \,\mathrm{erf}\left( r\sqrt{\frac{\mathrm{P}_c}{2}} \right) - \frac{\Pi \int_0^r \exp\left( -\frac{\mathrm{P}_c \zeta^2}{2} \right) \left[ \int_0^\zeta \frac{1}{\mu} \exp\left( \frac{\mathrm{P}_c \xi^2}{2} - 2\int_0^\xi \frac{\varepsilon}{\mu}\, d\varepsilon \right) d\xi \right] d\zeta}{\left[ \int_0^\infty \frac{1}{\mu} \exp\left( -\int_0^r \frac{\zeta}{\mu}\, d\zeta \right) dr \right]^2}. \tag{19}$$

If $A < 0$, then $\theta'(0) < 0$ and $\theta \leqslant 1$, so that the second boundary value for $\mu$ is $\mu_{\theta=1}$. If $A > 0$, then $\theta'(0) > 0$ and $\theta \leqslant 1 + A$. From formula (17), for each of these cases one can obtain an inequality of the form

$$f(m, M, \mathrm{P}_c, \Pi) \leqslant N \leqslant F(m, M, \mathrm{P}_c, \Pi), \tag{20}$$

and in the second case it must be considered together with the inequality $\theta \leqslant 1 + A$, which must be strengthened by introducing the numbers $m, M$ instead of $\mu$ in $A$ and $\theta(\mu)$, so that it takes the form

$$\Phi(m, M, \mathrm{P}_c, \Pi) \geqslant 0. \tag{21}$$

Analogous reasoning can be carried out for a liquid whose thermal conductivity is variable. For this it suffices to introduce a new unknown function

$$Q = \int_0^\theta \chi \, d\theta$$

and to consider $\mu = \mu(Q)$.

Received31 III 1956

## REFERENCES

1. S. M. Targ, *Basic Problems in the Theory of Laminar Flows*, Moscow–Leningrad, 1951.

*Note: Figure translations are in progress. See original paper for figures.*