ON THE REDUCIBILITY OF ONE CLASS OF SYSTEMS OF DIFFERENTIAL EQUATIONS WITH QUASIPERIODIC COEFFICIENTS

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Abstract

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MATHEMATICS

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ON THE REDUCIBILITY OF ONE CLASS OF SYSTEMS OF DIFFERENTIAL EQUATIONS WITH QUASIPERIODIC COEFFICIENTS

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The question of the reducibility ([1]) of systems of differential equations with almost periodic coefficients has received little attention in the literature; especially difficult is the case when zero is a point of accumulation of the frequencies of the coefficients. The principal difficulty distinguishing this question from the analogous question for periodic functions is apparently the fact that the family of almost periodic functions with zero mean value is not closed with respect to the operation of integration.

In this paper the question of reducibility is studied for one class of systems of two linear equations with quasiperiodic coefficients.

The investigation relies essentially on the relation, proved by N. P. Erugin ([1]), between the presence or absence of reducibility and the possibility of representing certain integrals in the form

$$\int \varphi(x)\,dx = at + \Phi(t), \tag{1}$$

where a is a constant; $\Phi(t)$ is a bounded function; $\varphi(t)$ is connected with the coefficients of the given system in a very complicated way ($\varphi(t)$ depends on the solution of a Riccati equation whose coefficients are linear combinations of the coefficients of the given system).

In the paper a new apparatus of corresponding majorant series is introduced, which makes it possible to estimate the coefficients of the Fourier series for the quasiperiodic solution of the aforementioned Riccati equation, and also for the function $\varphi(t)$, which allows one to verify the fulfillment of condition (1) and, consequently, to give a sufficient criterion for reducibility of the class of systems under study. It turns out here that the property of reducibility is essentially connected with the arithmetic nature of the frequencies of the coefficients of the system and therefore, generally speaking, possesses "instability" : an appropriately chosen arbitrarily small change of one of the frequencies makes a reducible system nonreducible.

Definition. Let $f(t)$ be a quasiperiodic function. Let

$$\bar{f}(\lambda) = \sum_{j=0}^{\infty} a_j \lambda^j,$$

where $a_j \geq 0$; $\lim_{j\to\infty} a_j = 0$.

We shall call $\bar{f}(\lambda)$ a corresponding majorant series (function) for the function $f(t)$ and write

$$f(t) \preccurlyeq \bar{f}(\lambda),$$

if

$$f(t) = \sum_{j=0}^{\infty} P_j(t),$$

and this series converges uniformly,

$$P_j(t) = \sum_{|m_1|+|m_2|+\cdots+|m_n|<j} \eta_{m_1 m_2 \cdots m_n} e^{it(m_1\omega_1+m_2\omega_2+\cdots+m_n\omega_n)*}, \qquad |P_j(t)| \leq a_j.$$

Let $f_1(t) \preccurlyeq \bar{f}_1(\lambda)$, $f_2(t) \preccurlyeq \bar{f}_2(\lambda)$.
Then Theorems 1-4 hold:

Theorem 1. $f_1(t) + f_2(t) \preccurlyeq \bar{f}_1(\lambda) + \bar{f}_2(\lambda)$.

Theorem 2. $f_1(t) \cdot f_2(t) \preccurlyeq \bar{f}_1(\lambda) \cdot \bar{f}_2(\lambda)$.

Theorem 3.

$$F(t) = e^{At} \int_t^{\infty} f_1(x) e^{-Ax}\, dx \preccurlyeq \frac{\bar{f}_1(\lambda)}{A} \qquad (A > 0).$$

Theorem 4. $|f_1(t)| \preccurlyeq \bar{f}_1(1)$ (if the series for $\bar{f}_1(\lambda)$ converges at $\lambda = 1$).

Theorem 5. Suppose a Riccati equation is given

$$\dot{\tau} = \xi(t) + f(t)\tau + r(t)\tau^2,$$

where $\xi(t)$, $f(t)$, $r(t)$ are quasiperiodic functions.

If there exist $\bar{\xi}(\lambda)$, $\bar{f}(\lambda)$, $\bar{r}(\lambda)$ (with radii of convergence of the series greater than 1) such that

$$\xi(t) \preccurlyeq \bar{\xi}(\lambda); \qquad f(t) \preccurlyeq \bar{f}(\lambda); \qquad r(t) \preccurlyeq \bar{r}(\lambda);$$

$$\bar{f}(1) + 2\sqrt{\bar{\xi}(1)\bar{r}(1)} < 2\bar{f}(0),$$

then:

1) there exists a quasiperiodic solution $\tau(t)$ of the equation;
2) there exists $\bar{\tau}(\lambda)$ such that the series corresponding to it has radius of convergence greater than unity and

$$\tau(t) \preccurlyeq \bar{\tau}(\lambda).$$

Theorem 6. Suppose

$$\varphi(t) \preccurlyeq \bar{\varphi}(\lambda),$$

and the radius of convergence of $\bar{\varphi}(\lambda)$ is greater than unity. Suppose

$$\int_0^t \varphi(x)\,dx = at + \Phi(t), \qquad \text{where } a = \lim_{l\to\infty} \frac{1}{l} \int_0^l \varphi(x)\,dx.$$

For the function $\Phi(t)$ to be bounded, it is sufficient that the series

$$\sum_{H=1}^{\infty} a_H, \qquad \text{where } a_H = \frac{H^{n-1} R_H[\bar{\varphi}(1)]}{\min\limits_{|m_1|+|m_2|+\dots+|m_n|=H} |m_1\omega_1 + m_2\omega_2 + \dots + m_n\omega_n|},$$

converge; by $R_H[\bar{\varphi}(\lambda)]$ is meant the remainder term of the series $\bar{\varphi}(\lambda)$ (beginning with degree λ^H).

Theorem 7[**]. In order that the function $\Phi(t)$ be bounded,

[*] $\omega_1, \omega_2, \dots, \omega_n$ are linearly independent numbers.
[**] Formulated in the notation of Theorem 6.

when the radius of convergence R of the function $\varphi(\lambda)$ is greater than unity, it is sufficient that

$$\overline{\lim}_{|m_1|+|m_2|+\dots+|m_n|=H\to\infty} |m_1\omega_1 + m_2\omega_2 + \dots + m_n\omega_n|^{-1/H} = r < R.$$

Theorem 8. Let the system

$$\dot{x} = P_{11}(t)x + P_{12}(t)y, \qquad \dot{y} = P_{21}(t)x + P_{22}(t)y, \tag{2}$$

be given, where $P_{11}(t), P_{12}(t), P_{21}(t), P_{22}(t)$ have radii of convergence of the corresponding majorant series greater than unity; suppose that the Riccati equation

$$\dot{\tau} = P_{12} + (P_{22} - P_{11})\tau - P_{21}\tau^2 \tag{3}$$

satisfies the conditions of Theorem 5, and let $R > 1$ be the least of the radii of convergence of the corresponding majorant series for $P_{11}(t)$, $P_{12}(t)$, $P_{21}(t)$, $P_{22}(t)$, $\tau(t)$.

Then, for the reducibility of system (2), it is sufficient that the inequality

$$\overline{\lim}_{|m_1|+|m_2|+\dots+|m_n|=H\to\infty} |m_1\omega_1 + m_2\omega_2 + \dots + m_n\omega_n|^{-1/H} = r < R$$

be fulfilled.

Theorem 9. Suppose that $P_{11}(t), P_{12}(t), P_{21}(t), P_{22}(t)$ in system (2) have radii of convergence of the corresponding majorant series greater than unity; suppose that the Riccati equation (3) satisfies the conditions of Theorem 5, and suppose that $\omega_1, \omega_2, \dots, \omega_n$ are algebraic numbers (or algebraic up to a common factor).

Then system (2) is reducible.

Example*.

$$\dot{x} = (-a + \sin\alpha t)x + y\cos\beta t, \qquad \dot{y} = x\sin\alpha t + y\sin^2\alpha t$$

(α and β are incommensurable, $a > 0$). Applying Theorems 5 and 8, we obtain the following results. If

$$\overline{\lim}_{|m_1|+|m_2|\to\infty} |m_1\alpha + m_2\beta|^{-1/(|m_1|+|m_2|)} < R,$$

where R is the least positive root of the equation

$$x + 2x^{3/2} + \frac{1}{2}x^2 = a + \frac{1}{2},$$

then the system is reducible. If α and β are algebraic incommensurable numbers, then the system is always reducible for $a > 3$.

It is shown in the paper that an arbitrarily small, suitably chosen change in the values of one of the frequencies of a quasiperiodic function generally entails the violation of boundedness of the nonlinear part of the indefinite integral of this function (and, consequently, the violation of quasiperiodicity of this part).

Hence follows the "instability" of the reducibility property for systems of the class under consideration.

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REFERENCES

N. P. Erugin, *Trudy Mat. Inst. im. V. A. Steklova*, **13** (1946).

* Proposed for study by N. P. Erugin.

Note: Figure translations are in progress. See original paper for figures.