# PHYSICS

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1957

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## Abstract

## Full Text

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# FINE STRUCTURE OF $\alpha$-DECAY OF EVEN-EVEN NUCLEI

*(Presented by Academician M. A. Leontovich on 3 XI 1956)*

As is known ([1,2]), in the $\alpha$-decay of nonspherical heavy nuclei excitation of the rotational levels of the daughter nucleus is observed. The behavior of the system daughter nucleus $+\alpha$-particle is described by the Schrödinger equation

$$\frac{\hbar^2}{2I}\mathbf{J}^2\Psi - \frac{\hbar^2}{2m}\nabla^2\Psi + U(\xi,\eta,\zeta)\Psi = E\Psi, \tag{1}$$

where $\mathbf{J}$ is the angular-momentum operator of the daughter nucleus, $I$ is its moment of inertia, $m$ is the reduced mass, $U$ is the potential energy of the electrostatic interaction of the $\alpha$-particle with the nucleus, and $E$ is the total energy of the system. The angular momentum of the daughter nucleus is expressed in terms of the total angular momentum of the system $\mathbf{K}$ and the orbital angular momentum of the $\alpha$-particle $\mathbf{l}$ by means of $\mathbf{J} = \mathbf{K} - \mathbf{l}$. Since for even-even nuclei (only such nuclei will be considered below) $\mathbf{K} = 0$, $\mathbf{J}^2 = \mathbf{l}^2$ and, consequently,

$$\frac{\hbar^2}{2I}\mathbf{l}^2\Psi - \frac{\hbar^2}{2m}\nabla^2\Psi + U(\xi,\eta,\zeta)\Psi = E\Psi. \tag{2}$$

This equation was first obtained by Strutinskii ([3]).

The potential energy depends only on the coordinates $\xi,\eta,\zeta$ of the $\alpha$-particle with respect to the nucleus, i.e., on the Cartesian coordinates in a reference system rotating together with the nucleus, and does not depend on the Euler angles $\Omega$, which determine the orientation of the rotating axes relative to the fixed axes. Therefore (2) is conveniently solved in the variables $\xi,\eta,\zeta,\Omega$.

On passing from the fixed coordinate system to the rotating one, the operators $\mathbf{l}^2$ and $\nabla^2$ retain their form. The wave function of the state with $K = 0$ does not depend on $\Omega$; passing from the Cartesian coordinates $\xi,\eta,\zeta$ to spherical $r, \mu = \cos\vartheta, \varphi$, it is also easy to see that it cannot depend on $\varphi$. As a result we obtain:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Psi) + \left(\frac{m}{I} + \frac{1}{r^2}\right)\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\Psi}{\partial\mu}\right] + \frac{2m}{\hbar^2}[E - U(r,\mu)]\Psi = 0. \tag{3}$$

A particular solution describing an $\alpha$-particle with angular momentum $l$, moving away from the nucleus, has the form:

$$\Psi_l(r,\mu) = \frac{\sqrt{2l+1}}{2} P_l(\mu)\psi_l(r,\mu), \tag{4}$$

where $\psi_l$ ceases to depend on $\mu$ at large $r$ (the factor $\frac{\sqrt{2l+1}}{2}$ is introduced for convenience). Substitution into (3) leads to the equation

$$P_l\left\{\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi_l) + \left[\frac{2m}{\hbar^2}(E-U) - l(l+1)\left(\frac{m}{I}+\frac{1}{r^2}\right)\right]\psi_l\right\} +$$

$$+\left(\frac{m}{I}+\frac{1}{r^2}\right)\left\{P_l\left[(1-\mu^2)\frac{\partial^2\psi_l}{\partial\mu^2} - 2\mu\frac{\partial\psi_l}{\partial\mu}\right] + 2(1-\mu^2)\frac{dP_l}{d\mu}\frac{\partial\psi_l}{\partial\mu}\right\} = 0, \tag{5}$$

which is convenient to solve in the quasiclassical approximation:

$$\psi_l = e^{\sigma},$$

$$\begin{gathered} P_l\left\{\left(\frac{\partial\sigma}{\partial r}\right)^2 + \frac{2}{r}\frac{\partial\sigma}{\partial r} + \frac{\partial^2\sigma}{\partial r^2} + \frac{2m}{\hbar^2}(E-U) - l(l+1)\left(\frac{m}{I}+\frac{1}{r^2}\right)\right\} + \\ +\left(\frac{m}{I}+\frac{1}{r^2}\right)\left\{P_l\left[(1-\mu^2)\left(\frac{\partial\sigma}{\partial\mu}\right)^2 - 2\mu\frac{\partial\sigma}{\partial\mu} + (1-\mu^2)\frac{\partial^2\sigma}{\partial\mu^2}\right]\right. \\ \left. +2(1-\mu^2)\frac{dP_l}{d\mu}\frac{\partial\sigma}{\partial\mu}\right\} = 0. \end{gathered} \tag{6}$$

The corresponding expansion for $\sigma(r,\mu)$ will be written in the form $\sigma = \sigma_{-1} + \sigma_0 + \sigma_1 + ...$; for the successive approximations $\sigma_k$ we have an infinite sequence of equations

$$\left(\frac{\partial\sigma_{-1}}{\partial r}\right)^2 + \left(\frac{m}{I}+\frac{1}{r^2}\right)(1-\mu^2)\left(\frac{\partial\sigma_{-1}}{\partial\mu}\right)^2 = -\frac{2m}{\hbar^2}(E-U) + l(l+1)\left(\frac{m}{I}+\frac{1}{r^2}\right);$$

$$\begin{gathered} P_l\left[2\frac{\partial\sigma_{-1}}{\partial r}\frac{\partial\sigma_0}{\partial r} + \frac{2}{r}\frac{\partial\sigma_{-1}}{\partial r} + \frac{\partial^2\sigma_{-1}}{\partial r^2}\right] + \left(\frac{m}{I}+\frac{1}{r^2}\right)\left\{P_l\left[2(1-\mu^2)\frac{\partial\sigma_{-1}}{\partial\mu}\frac{\partial\sigma_0}{\partial\mu}\right.\right. \\ \left.\left. -2\mu\frac{\partial\sigma_{-1}}{\partial\mu} + (1-\mu^2)\frac{\partial^2\sigma_{-1}}{\partial\mu^2}\right] + 2(1-\mu^2)\frac{dP_l}{d\mu}\frac{\partial\sigma_{-1}}{\partial\mu}\right\} = 0, \end{gathered} \tag{7}$$

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The solution of system (7) is simplified in the case of a weakly nonspherical daughter nucleus. In this case the equation of the nuclear surface is conveniently represented in the form

$$R(\mu) = R_0 + \xi(\mu) = R_0 \left\{ 1 + \sum_n \alpha_n P_n(\mu) \right\}. \tag{8}$$

The quantities $\sigma_k$ are then sought in the form of series in powers of $\xi$: $\sigma_k = \sigma_k^{(0)} + \sigma_k^{(1)} + \sigma_k^{(2)} + \ldots$. The electrostatic potential produced by the daughter nucleus is easily calculated in the form of an analogous series (it is assumed that the nucleus is uniformly charged throughout its volume). As a result of simple calculations we obtain:

$$\sigma_{-1}^{(0)} = i \int_{a_l}^{r} k_l \, dr, \qquad \sigma_0^{(0)} = -\ln \sqrt{-i k_l r},$$

$$\sigma_{-1}^{(1)} = \frac{3}{2} i \chi_b^2 \sum_n \frac{\alpha_n}{2n+1} P_n(\mu) \int_r^{\infty} \left( \frac{R_0}{r} \right)^{n+1} \frac{dr}{k_l}. \tag{9}$$

Here

$$k_l = \sqrt{k_\varepsilon^2 - \chi_b^2 \frac{R_0}{r} - \frac{l(l+1)}{r^2}}, \qquad \chi_b^2 = \frac{4mZe^2}{\hbar^2 R_0},$$

$$k_\varepsilon^2 = \frac{2m\varepsilon}{\hbar^2}, \qquad \varepsilon \equiv \varepsilon_l = E - \frac{\hbar^2}{2I} l(l+1).$$

The turning point $a_l$ is determined from the equation

$$k_l^2(a_l) = k_\varepsilon^2 - \chi_b^2 \frac{R_0}{a_l} - \frac{l(l+1)}{a_l^2} = 0.$$

Formulas (9) are also valid for $r < a_l$, if one takes into account that in this region

$$k_l = i\chi_l = i \sqrt{\frac{l(l+1)}{r^2} + \chi_b^2 \frac{R_0}{r} - k_\varepsilon^2}.$$

The further terms of the double series $\sigma = \sum_{k,m} \sigma_k^{(m)}$ are negligibly small.

Substituting (9) into (6) and (4) and passing to $r \to \infty$, we obtain:

$$\Psi_l \to \frac{\sqrt{2l+1}}{2} \frac{1}{\sqrt{-ik_\varepsilon r}} \left[ \exp i \left( k_\varepsilon r - \frac{x_b^2 R_0}{2k_\varepsilon} \ln 2k_\varepsilon r + \delta_\varepsilon \right) \right] P_l(\mu),$$

i.e., the corresponding flux through a closed sphere does not depend on $l$. Therefore, if the wave function of the system is represented in the form $\Psi = \sum_l c_l \Psi_l$, then the relative probability of excitation of the rotational level with angular momentum $l$ will be equal to $w_l = |c_l|^2$.

It is not difficult to express the coefficients $c_l$ in terms of the value of the wave function on the nuclear surface $\Psi(S)$. Indeed, since the values of $l$ of interest to us are small, in the required approximation we have

$$\begin{gathered} \sigma_{-1}^{(0)}(S) = \sigma_{-1}^{(0)}(R_0)\big|_{l=0} - \varkappa_0(R_0)\xi + \frac{\gamma}{2} l(l+1), \\ \sigma_{-1}^{(1)}(S) = \sigma_{-1}^{(1)}(R_0)\big|_{l=0}, \qquad \sigma_0^{(0)}(S) = \sigma_0^{(0)}(R_0)\big|_{l=0}, \end{gathered} \tag{10}$$

where

$$\gamma = 2 \frac{\partial \sigma_{-1}^{(0)}}{\partial l(l+1)} (R_0) \bigg|_{l=0}.$$

Simple calculations lead to the formula

$$\gamma = \frac{2\varkappa}{x_b^2 R_0} + \frac{m/I}{k^2} \left( \frac{x_b^2 R_0}{k} \operatorname{arctg} \frac{\varkappa}{k} + \varkappa R_0 \right), \tag{11}$$

where

$$k = k_E = \sqrt{\frac{2mE}{\hbar^2}}, \qquad \varkappa = \varkappa_0(R_0) = \sqrt{x_b^2 - k^2}.$$

Substituting (10) into (6) and (4), we obtain:

$$\Psi(S) = \sum_l c_l \Psi_l(S) =$$

$$= \sum_l c_l \frac{\sqrt{2l+1}}{2} \frac{1}{\sqrt{\varkappa R_0}} \left[ \exp \left( \sigma_{-1}^{(0)}\big| - \varkappa\xi + \frac{\gamma}{2} l(l+1) + \sigma_{-1}^{(1)}\big| \right) \right] P_l(\mu). \tag{12}$$

In the last sum, $r = R_0$ is understood everywhere. The vertical line to the right of the corresponding expression means that it should be taken at $l = 0$. The factor independent of $l$,

$$\frac{1}{\sqrt{\varkappa R_0}} e^{\sigma_{-1}^{(0)}| - \varkappa\xi + \sigma_{-1}^{(1)}|}$$

can be taken outside the summation sign and transferred to the other side of the equality. As a result, application of the completeness theorem gives:

$$c_l = \sqrt{\varkappa R_0}\, e^{-\sigma_{-1}^{(0)}|} \sqrt{2l+1}\, e^{-\frac{\gamma}{2} l(l+1)} \int_{-1}^{1} \Psi(S) e^{\varkappa\xi - \sigma_{-1}^{(1)}|} P_l(\mu)\, d\mu,$$

$$w_l = \varkappa R_0^2 e^{-2\sigma_{-1}^{(0)}|} (2l+1) e^{-\gamma l(l+1)} \left| \int_{-1}^{1} \Psi(S) e^{\varkappa\xi - \sigma_{-1}^{(1)}|} P_l(\mu)\, d\mu \right|^2 . \qquad (13)$$

In the case of weakly nonspherical nuclei, it seems most natural to assume $\Psi(S) = \text{const}$. Normalizing the relative probability $w_l$ by the condition $w_0 = 1$, we finally obtain:

$$w_l = (2l+1) e^{-\gamma l(l+1)} \left| \frac{\int_{-1}^{1} e^{\varkappa\xi - \sigma_{-1}^{(1)}|} P_l(\mu)\, d\mu}{\int_{-1}^{1} e^{\varkappa\xi - \sigma_{-1}^{(1)}|}\, d\mu} \right|^2 . \qquad (14)$$

As is known ([4]), in heavy nuclei the most significant is a quadrupole deformation of the form $\xi = R_0 \alpha_2 P_2(\mu)$. Substitution into (9) and (14) gives

$$w_l = (2l+1) e^{-\gamma l(l+1)} \left| \frac{\int_{0}^{1} e^{\beta P_2(\mu)} P_l(\mu)\, d\mu}{\int_{0}^{1} e^{\beta P_2(\mu)}\, d\mu} \right|^2 , \qquad (15)$$

where

$$\beta = \left[ \frac{4}{5} \varkappa R_0 \left( 1 - \frac{k^2}{2\varkappa_b^2} \right) - i \frac{2}{5} \frac{k^3 R_0}{\varkappa_b^2} \right] \alpha_2. \qquad (16)$$

The physical meaning of (15) is simple. The exponential factor represents the dependence of the penetrability of the Coulomb barrier on the energy and angular momentum of the $\alpha$-particle. In particular, it takes into account the nonadiabatic character of the process, i.e., the circumstance that during the passage of the $\alpha$-particle through the subbarrier region the nucleus has time to rotate through an appreciable angle. Therefore the results obtained earlier in the adiabatic approximation ([5,6]) are not entirely reliable. The integral under the modulus squared sign describes the direct influence of the deformation of the nuclear surface $\xi(\mu)$, supplemented by the action of the quadrupole potential $\sigma_{-1}^{(1)}$. We note that this integral can be easily calculated with the aid of tables ([7]) of the function

$$w(z) = e^{-z^2} \int_0^z e^{x^2}\, dx.$$

**Table 1**

| Nucleus | $\alpha_2$ (I) | $\alpha_2$ (II) | $Q_0 = \frac{6}{5} Z R_0^2 \alpha_2 \cdot 10^{24}$ (I) | $Q_0 = \frac{6}{5} Z R_0^2 \alpha_2 \cdot 10^{24}$ (II) | Nucleus | $\alpha_2$ (I) | $\alpha_2$ (II) | $Q_0 = \frac{6}{5} Z R_0^2 \alpha_2 \cdot 10^{24}$ (I) | $Q_0 = \frac{6}{5} Z R_0^2 \alpha_2 \cdot 10^{24}$ (II) |
|---|---|---|---|---|---|---|---|---|---|
| $Em^{220}$ | 0.26 | 0.21 | 10 | 16 | $U^{230}$ | 0.13 | 0.11 | 5 | 9 |
| $Em^{222}$ | 0.29 | 0.23 | 11 | 17 | $U^{232}$ | 0.15 | 0.13 | 6 | 11 |
| $Ra^{222}$ | 0.24 | 0.20 | 9 | 15 | $U^{234}$ | 0.20 | 0.16 | 8 | 13 |
| $Ra^{224}$ | 0.30 | 0.24 | 12 | 18 | $U^{236}$ | 0.18 | 0.15 | 8 | 12 |
| $Ra^{226}$ | 0.26 | 0.21 | 10 | 16 | $U^{238}$ | 0.16 | 0.14 | 7 | 12 |
| $Ra^{228}$ | 0.30 | 0.24 | 12 | 19 | $Pu^{238}$ | 0.18 | 0.15 | 8 | 13 |
| $Th^{226}$ | 0.27 | 0.22 | 11 | 17 | $Pu^{240}$ | 0.17 | 0.15 | 7 | 13 |
| $Th^{228}$ | 0.26 | 0.21 | 10 | 17 | $Cm^{242}$ | 0.15 | 0.13 | 7 | 11 |
| $Th^{230}$ | 0.21 | 0.17 | 9 | 14 | $Cm^{246}$ | 0.13 | 0.11 | 6 | 10 |
| $Th^{232}$ | 0.23 | 0.19 | 9 | 15 | $Cm^{248}$ | 0.12 | 0.10 | 5 | 9 |
| $Th^{234}$ | 0.21 | 0.17 | 9 | 14 | $Cf^{250}$ | 0.12 | 0.10 | 6 | 9 |

It appears possible, on the basis of experimental data ([1,8,9]), to calculate from formulas (11), (15), and (16) the deformation $\alpha_2$ of some nuclei. The quantities $k$, $I$, $w_2$ are determined experimentally; for the nuclear radius two alternative values must be taken: $R_0 = 1.0 A^{1/3} \cdot 10^{-13}$ cm (I) and $R_0 = 1.4 A^{1/3} \cdot 10^{-13}$ cm (II). The results of the treatment of the experimental data are presented in Table 1. The calculated quadrupole moment $Q_0$ of the daughter nucleus can be checked by measuring the cross section of Coulomb excitation $0_+ \to 2_+$ or the probability of emission of the $\gamma$-quantum $2_+ \to 0_+$ ([4]).

Received
29 VIII 1956

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*Note: Figure translations are in progress. See original paper for figures.*