Mathematics

1957

SovietRxiv

Abstract

Full Text

Mathematics

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Secant Surfaces of Double Fibrations

(Presented by Academician P. S. Aleksandrov on 20 X 1956)

Let $\mathfrak{F}_1 = \{P_1, p, B\}$ and $\mathfrak{F}_2 = \{P_2, p', P_1\}$ be two fiber spaces in the sense of Serre (1), the base of $\mathfrak{F}_1$ being a simply connected simplicial complex B. The fibers $C_1 = p^{-1}(x_0)$ and $C' = (p')^{-1}(*)$ (where $x_0 \in B$, $* \in C_1 \subset P_1$) are assumed to be homotopically simple in dimensions r and $(r-1)$, respectively. Suppose that over an r-dimensional skeleton B^r of the base space B, two secant surfaces $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are given in $\mathfrak{F}_1$, coinciding over the skeleton B^{r-1}.

The mapping $\mathfrak{S}_i : B^r \to P_1$ $(i = 1, 2)$ and the fiber space $\mathfrak{F}_2$ induce the fiber space

$$\mathfrak{F}_{\mathfrak{S}i} = \{P_{\mathfrak{S}i}, p_i, B^r\}$$

with base B^r. Here $P_{\mathfrak{S}i}$ is the subspace of the direct product $B^r \times P_2$ consisting of all pairs (x, g) satisfying the condition $\mathfrak{S}_i(x) = p'(g)$, and the projection p_i is defined by the formula $p_i(x, g) = x$.

Since $\mathfrak{F}_{\mathfrak{S}1}$ and $\mathfrak{F}_{\mathfrak{S}2}$ coincide over B^{r-1}, any secant surface ψ, given in $\mathfrak{F}_{\mathfrak{S}1}$ on B^{r-1}, may also be regarded as a secant surface in $\mathfrak{F}_{\mathfrak{S}2}$ on B^{r-1}. We shall assume the fiber spaces $\mathfrak{F}_1, \mathfrak{F}_2$ and the secant surfaces $\mathfrak{S}_1, \mathfrak{S}_2$ to be such that, in the fiber spaces $\mathfrak{F}_{\mathfrak{S}1}$ and $\mathfrak{F}_{\mathfrak{S}2}$, a secant surface can be constructed over the entire $(r-1)$-dimensional skeleton B^{r-1}. Fix one such secant surface (on B^{r-1}) and henceforth denote it by ψ.

Denote by $z^r_{1,\psi}$ and $z^r_{2,\psi}$ the obstructions to extending the secant surface ψ in the fiber spaces $\mathfrak{F}_{\mathfrak{S}1}$ and $\mathfrak{F}_{\mathfrak{S}2}$, and by Z^r_1 and Z^r_2 the cohomology classes* of these obstructions. Here Z^r_1 and Z^r_2 are elements of the cohomology group $H^r(B^r, \pi^{r-1}(C'))$.

Next denote by $d^r_{\mathfrak{S}1,\mathfrak{S}_2}$ the difference cochain of the secant surfaces $\mathfrak{S}_1$ and $\mathfrak{S}_2$. We may regard this cochain as an r-dimensional cocycle of the complex B^r. The cohomology class of this cocycle shall be denoted by $D^r_{\mathfrak{S}1,\mathfrak{S}_2}$; this class is an element of the group

$$H^r(B^r, \pi^r(C_1)).$$

Finally, put $C_2 = (p')^{-1}(C_1)$ and denote by p' again the mapping p' considered on C_2. Then $\{C_2, p', C_1\}$ is a fiber space (a part of the fiber space $\mathfrak{F}_2$) with fiber $C' = (p')^{-1}(*)$.

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* Following a proposal of V. G. Boltyansky, we shall use the term cohomology instead of the previously used upper homology, ∇-homology, cohomology, since this term better corresponds to the nature of the concept under consideration. Accordingly, we shall speak of cochains, cocycles, etc.

For this fiber space we can write the exact homotopy sequence

$$\ldots \longrightarrow \pi^r(C_2) \xrightarrow{p'_*} \pi^r(C_1) \xrightarrow{\Delta} \pi^{r-1}(C') \longrightarrow \pi^{r-1}(C_2) \longrightarrow \ldots, \tag{1}$$

where Δ is the boundary homomorphism. This homomorphism

$$\Delta: \quad \pi^r(C_1) \longrightarrow \pi^{r-1}(C')$$

induces a homomorphism of cohomology groups

$$H^r(B^r, \pi^r(C_1)) \longrightarrow H^r(B^r, \pi^{r-1}(C')),$$

which we shall denote by $\hat{\Delta}$.

Under the stated assumptions the following theorem holds.

Theorem. *The cohomology classes* $Z_1^r, Z_2^r, D^r_{\mathfrak{S}_1, \mathfrak{S}_2}$ *are related by*

$$Z_1^r - Z_2^r = \hat{\Delta} D^r_{\mathfrak{S}_1, \mathfrak{S}_2}. \tag{2}$$

An essential role in the proof of this theorem is played by the following construction. Let α be an element of the homotopy group $\pi^r(C_1)$, and let $f : S^r \to C_1$ be a map of the oriented sphere S^r which takes some point $y \in S^r$ to the point $*$ and belongs to the class α. The map f and the fiber space $\{C_2, p', C_1\}$ induce a new fiber space with base S^r and fiber C'. Denote by $Z^r \in H^r(S^r, \pi^{r-1}(C'))$ the characteristic cochain (i.e. the cohomology class of the first obstruction) of this fiber space, and by $\beta \in \pi^{r-1}(C')$ the index of the cohomology class Z^r on the oriented sphere S^r. It is easily established that the element β does not depend on the choices made in the construction and is uniquely determined by the element α, so that one may set $\beta = \chi(\alpha)$.

Lemma. *The map defined above*

$$\chi: \quad \pi^r(C_1) \longrightarrow \pi^{r-1}(C')$$

coincides with the boundary homomorphism Δ *of the exact sequence (1).*

Let us outline, in its main features, the proof of formula (2). Let T^r be an arbitrary r-dimensional oriented simplex of the complex B, and let T^+ and T^-

be two identical copies of it, glued along their boundaries. We orient the sphere $S^r = T^+ \cup T^-$ consistently with T^+. The map

$$\mathfrak{S}(x) = \begin{cases} \mathfrak{S}_1(x), & \text{if } x \in T^+, \\ \mathfrak{S}_2(x), & \text{if } x \in T^- \end{cases}$$

of the sphere S^r into P_1 and the space $\mathfrak{P}_2$ induce over S^r a fiber space $\mathfrak{P}_{\mathfrak{S}} = \{P_{\mathfrak{S}}, p, S^r\}$, which, evidently, on T^+ coincides with the part of the space $\mathfrak{P}_{\mathfrak{S}_1}$ over T^r, and on T^- with the part of $\mathfrak{P}_{\mathfrak{S}_2}$ over T^r.

Let k'_t be a deformation joining the identity map $k'_0 : T^r \to B$ with the map k'_1 that sends T^r to the point x_0. Put $k_t = k'_t \circ e$, where $e : S^r \to T^r$ maps each of the simplexes T^+, T^- identically onto T^r. From $p \circ \mathfrak{S} = e = k_0$, applying the covering homotopy existence condition to the fiber space $\mathfrak{P}_1$, we find a deformation $\mathfrak{S}^t$ of the map $\mathfrak{S}^0 = \mathfrak{S}$ such that $p \circ \mathfrak{S}^t = k_t$. The map $\mathfrak{S}^1$ carries the sphere S^r into the fiber C_1 and determines an element $d^r_{\mathfrak{S}_1,\mathfrak{S}_2}(T^r)$ of the group $\pi^r(C_1)$, i.e. the value of the difference $d^r_{\mathfrak{S}_1,\mathfrak{S}_2}$ on the simplex T^r.

The map $\mathfrak{S}^t : S^r \to P_1$ and the fibered space $\mathfrak{P}_2$ induce a new fibered space

$$\mathfrak{P}_{\mathfrak{S}^t} = \{P_{\mathfrak{S}^t}, p'_t, S^r\};$$

the points of the space $P_{\mathfrak{S}^t}$ are pairs (x, g), $x \in S^r$, $g \in P_2$, satisfying the condition

$$\mathfrak{S}^t(x) = p'(g).$$

Define the map $h^t : P_{\mathfrak{S}^t} \to P_2$ by setting

$$h^t(x, g) = g.$$

The map ψ is defined on the whole base B^{r-1} and, in particular, on the sphere $S^{r-1} = T^r$. Since $\mathfrak{P}_{\mathfrak{S}_1} = \mathfrak{P}_{\mathfrak{S}_2}$ over B^{r-1}, the map ψ, considered on S^{r-1}, may be regarded as a section surface of the fibered space $\mathfrak{P}_{\mathfrak{S}^0}$ defined on S^{r-1}. This map

$$S^{r-1} \to P_{\mathfrak{S}^0}$$

will, for convenience, be denoted by ψ^0. The map

$$\varphi^0 = h^0 \circ \psi^0 : \quad S^{r-1} \to P_2$$

obviously satisfies the condition

$$p' \circ \varphi^0 = \mathfrak{S}^0,$$

and, by the condition for the existence of a covering homotopy, one can find a continuous family of maps

$$\varphi^t : \quad S^{r-1} \to P_2,$$

such that (on S^{r-1})

$$p' \circ \varphi^t = \mathfrak{S}^t.$$

Now put

$$\psi^t(x) = (x, \varphi^t(x)), \quad x \in S^{r-1},$$

and we obtain a section surface ψ^t of the fibered space $P_{\mathfrak{S}^t}$, defined over S^{r-1}. In particular, we obtain the section surface

$$\psi^1 : \quad S^{r-1} \to P_{\mathfrak{S}^1}.$$

The obstruction (in the fibered space $\mathfrak{P}_{\mathfrak{S}^1}$) to extending this section surface has, on the cells T^+ and T^- (oriented in the same way as T^r), certain values z^+ and z^- (which are elements of the group $\pi^{r-1}(C')$; here it is assumed that the map $\mathfrak{S}^1$ sends some point of the sphere S^{r-1} to the point $*$, which entails no loss of generality). According to the lemma formulated above, we have

$$z^+ - z^- = \Delta d^r_{\mathfrak{S}_1, \mathfrak{S}_2}(T^r). \tag{3}$$

It is not difficult to show (by constructing a connecting deformation depending on two parameters) that the elements z^+ and z^- do not depend on the choice of the auxiliary deformations $k'_t, \mathfrak{S}^t$ and φ^t, but are uniquely determined by the choice of the section surfaces $\mathfrak{S}_1, \mathfrak{S}_2, \psi$ and the oriented simplex T^r. Choosing the auxiliary deformations in a special way, it is not difficult to verify that

$$z^+ = z^r_{1,\psi}(T^r), \quad z^- = z^r_{2,\psi}(T^r). \tag{4}$$

Formula (2) follows from (3), (4).

Let us consider some special cases of the theorem proved. Let $\mathfrak{P}_1 = \{P_1, p_1, B, C_1, \mathfrak{G}\}$ and $\mathfrak{P}_2 = \{P_2, p_2, B, C_2, \mathfrak{G}\}$ be two skew products (^{2}) with the same coordinate transformations.

for which the base is one and the same simply connected simplicial complex B; the fibers have the form $C_1 = \mathcal{G}/\Gamma_1$, $C_2 = \mathcal{G}/\Gamma_2$, where $\mathcal{G}$ is a transitive group of transformations of the fibers C_1, C_2, and Γ_1, Γ_2 are such stable subgroups of $\mathcal{G}$ that the inclusion $\mathcal{G} \supset \Gamma_1 \supset \Gamma_2$ holds.

There arises a natural mapping p' (by inclusion):

$$p' : \quad C_2 = \mathcal{G}/\Gamma_2 \to C_1 = G/\Gamma_1,$$

and, in consequence of the coincidence of the coordinate transformations, the mapping

$$p' : \quad P_2 \to P_1.$$

If the subgroup Γ_2 has a local secant surface, then this mapping gives a new skew product with fiber $C' = \Gamma_1/\Gamma_2$. If the same restrictions as before are imposed on the spaces B, C_1, C', then the conditions of applicability of the theorem proved above will be satisfied. In this case the homomorphism Δ of the exact sequence (1) can be included in the following commutative diagram, which facilitates its computation:

$$\begin{tikzcd}
\pi^r(\mathcal{G}, \Gamma_1) \arrow[r, "p_*"] \arrow[dd, "\partial"'] & \pi^r(\mathcal{G}/\Gamma_1) \arrow[dd, "\Delta"'] \arrow[dr, "(p'_*)^{-1}"] & \\
 & & \pi^r(\mathcal{G}/\Gamma_2, \Gamma_1/\Gamma_2). \\
\pi^{r-1}(\Gamma_1) \arrow[r, "\bar{p}_*"] & \pi^{r-1}(\Gamma_1/\Gamma_2) \arrow[ur, "\partial"'] &
\end{tikzcd} \tag{5}$$

Let us consider an example.

Let $\mathcal{P}_1 = \{P_1,\ p_1,\ M^n,\ C_1,\ \mathcal{G}\}$, $\mathcal{P}'_2 = \{P_2,\ p_2,\ M^n,\ C_2, \mathcal{G}\}$ be two skew products whose base is a Riemannian n-dimensional manifold M^n, oriented and triangulated, and whose spaces P_1 and P_2 consist of all k-frames and, respectively, $(k+1)$-frames tangent to the manifold M^n. The fiber C_1 in $\mathcal{P}_1$ is the Stiefel manifold $C_1 = V_{n,k} = SO(n)/SO(r)$ $(k = n - r)$, and the fiber C_2 in $\mathcal{P}'_2$ is $C_2 = V_{n,k+1} = SO(n)/SO(r-1)$. It is clear that such fibers and the group $\mathcal{G} = SO(n)$ satisfy the required conditions.

Since $SO(n) \supset SO(r) \supset SO(r-1)$, there arises a natural mapping (discarding the last vector of the frame)

$$p' : \quad C_2 = V_{n,k+1} \to C_1 = V_{n,k}.$$

We obtain the skew product

$$\mathcal{P}_2 = \{V_{n,k+1},\ p',\ V_{n,k},\ S^{r-1},\ SO(r)/SO(r-1)\}.$$

In view of the triviality of the groups $\pi^s(S^{r-1})$ for $s < r - 1$, we can construct a secant surface ψ over B^{r-1} in $\mathcal{P}_{\mathfrak{S}_i}$.

In the present case it is not difficult to compute the homomorphisms $\partial, \bar{p}_*$ by means of the results of the computation of the groups $\pi^r(V_{n,k})$.

I take this occasion to express my gratitude to V. G. Boltyanskii, under whose direction the present work was carried out.

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Received
19 X 1956

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Note: Figure translations are in progress. See original paper for figures.