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Abstract

Full Text

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THE EQUATION OF HYDROSTATIC EQUILIBRIUM FOR SPHERICAL STELLAR SYSTEMS

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1. Let ρ be the density; p, the pressure; r, the radius vector with respect to the center of mass; $U(r)$, the gravitational potential. Then the hydrostatic equation for spherical fluid masses has the form

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{dU}{dr}. \tag{1}$$

The well-known Emden equation is obtained from equation (1) when a polytropic equation of state is prescribed,

$$p = K\rho^{\gamma}, \tag{2}$$

where K and γ are constant coefficients.

The Emden equation was used as early as 1911 by Plummer ([1]) to determine the distribution of stars in globular clusters and, as is known, a particular solution of this equation corresponding to $\gamma = 6/5$ (the so-called "Schuster law") sometimes gives a fair representation of observational results. Subsequently, the legitimacy of applying to stellar systems analogies with molecular gas, on which the use of the Emden equation is based, was repeatedly subjected to serious criticism by Jeans ([2]), Eddington ([3]), and others. Recently this question was again considered by van der Wulli ([4]).

It is therefore of interest to give a derivation of the equation of equilibrium of a spherical stellar system, avoiding, as far as possible, any artificial hypotheses about the properties of stellar systems.

2. We shall consider a stellar system: 1) which is in a stationary state; 2) which has spherical symmetry in the distribution of stellar density; 3) in addition, we shall make only one assumption, namely that for it the phase density is defined:

$$f = f(r; R, \Theta, \Phi), \tag{3}$$

which satisfies Boltzmann' s equation. Here r is, as before, the radius vector; R, Θ, Φ are Jeans' s spherical components of the linear velocity:

$$R = \dot{r}, \qquad \Theta = r\dot{\theta}, \qquad \Phi = r\sin\theta\,\dot{\varphi}, \tag{4}$$

where θ and φ are the angular spherical coordinates, and a dot above a letter denotes differentiation with respect to time.

The phase density must satisfy the obvious conditions: $f \geq 0$, and, moreover, the product of f by any integral polynomial of the second degree $P(R, \Theta, \Phi)$ tends to zero if at least one of the velocity components tends to infinity.

In our case the Boltzmann equation has the form

$$R\frac{\partial f}{\partial r}+\left\{\frac{dU}{dr}+\frac{\Theta^2+\Phi^2}{r}\right\}\frac{\partial f}{\partial R}+\left\{\operatorname{cotg}\theta\,\frac{\Phi^2}{r}-\frac{R\Theta}{r}\right\}\frac{\partial f}{\partial \Theta}+\left\{-\operatorname{cotg}\theta\,\frac{\Theta\Phi}{r}-\frac{\Phi R}{r}\right\}\frac{\partial f}{\partial \Phi}=0. \tag{5}$$

Using the condition that f must not depend on the angular coordinate θ, we can decompose equation (5) into two equations:

$$\Phi\frac{\partial f}{\partial \Theta}-\Theta\frac{\partial f}{\partial \Phi}=0;$$

$$R\frac{\partial f}{\partial r}+\left\{\frac{dU}{dr}+\frac{\Theta^2+\Phi^2}{r}\right\}\frac{\partial f}{\partial R}-\frac{R}{r}\left\{\Theta\frac{\partial f}{\partial \Theta}+\Phi\frac{\partial f}{\partial \Phi}\right\}=0. \tag{6}$$

The first of these equations means that f must depend only on the quantity

$$T^2 = \Theta^2 + \Phi^2, \tag{7}$$

i.e., on the square of the transverse component of the velocity T. Then the second equation can be written in the form

$$R\frac{\partial f}{\partial r}+\left\{\frac{dU}{dr}+\frac{T^2}{r}\right\}\frac{\partial f}{\partial R}-\frac{RT}{r}\frac{\partial f}{\partial T}=0, \tag{8}$$

which was first obtained by Schwarzschild[(5)].

To obtain the equation of hydrostatic equilibrium, we multiply equation (8) by R and integrate over the entire velocity space. Denoting the volume element in velocity space by $d\Omega$, we introduce the notation:

$$\nu = \int f\,d\Omega, \qquad p = \int R^2 f\,d\Omega, \qquad q = \int T^2 f\,d\Omega, \tag{9}$$

and then obtain the general equation of hydrostatic equilibrium for spherical stellar systems

$$\frac{dp}{dr} + \frac{2p - q}{r} = \nu \frac{dU}{dr}. \tag{10}$$

Here ν is the stellar density; p and q are "pressure" parameters analogous to the Jeans parameters introduced by him in 1922 for the case of systems with axial symmetry.

Despite its simplicity, equation (10) has until now remained unknown, although from it one can derive a number of general properties of stellar systems.

3. The equation of hydrostatic equilibrium for a fluid (1) is a special case of equation (10), corresponding to the case when

$$2p - q = 0. \tag{11}$$

The parameters p and q depend on the law of distribution of stellar velocities at each point of the stellar system and can be written in the form

$$p = \nu \overline{R^2}, \qquad q = \nu \left(\overline{\Theta^2} + \overline{\Phi^2} \right). \tag{12}$$

It is evident from this that condition (11) is satisfied if $\overline{R^2} = \overline{\Theta^2} = \overline{\Phi^2}$, i.e., if the velocity distribution has spherical symmetry.

In the general case (when $2p - q \neq 0$), equation (10) contains four unknown functions. Of these, two—ν and $U(r)$—are connected by Poisson' s equation. Hence it is evident that in the general case the problem is indeterminate, and it is necessary to specify two more relations between these functions in order for a unique solution to become possible.

In the Emden—Plummer case, uniqueness is achieved by introducing condition (11) and the equation of state (polytrope) (2).

From the preceding it is clear that, in order to solve the problem of hydrostatic equilibrium, it is necessary to find a relation between essentially statistical elements, namely p, q, and partly ν. In other words, the solution of the hydrodynamic problem is impossible in the present case without bringing in the methods of statistical mechanics.

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Note: Figure translations are in progress. See original paper for figures.