MATHEMATICS

K. K. BILEVICH

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Abstract

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K. K. BILEVICH

ON THE IDENTITY OF TWO ALGEBRAIC FIELDS OF THE n-TH ORDER

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§ 1. **Regions $(1)^{(k)}$ and sequences $\{1\}^{(k)}$ of points of the lattice $[\omega]$, repeated by multiplication.** Consider, in the n-dimensional complex space K_n, an n-dimensional lattice, repeated by multiplication, with basis points $\omega_1, \omega_2, \dots, \omega_n$. Suppose that the first r coordinates $\omega_k^{(1)}, \omega_k^{(2)}, \dots, \omega_k^{(r)}$ of every basis point ω_k are real, while the remaining $n - r = 2t$ coordinates are complex numbers:

$$\omega_k^{(r+1)} = \rho_k^{(1)} + i\sigma_k^{(1)}, \quad \omega_k^{(r+2)} = \rho_k^{(1)} - i\sigma_k^{(1)}, \dots, \quad \omega_k^{(n-1)} = \rho_k^{(t)} + i\sigma_k^{(t)}, \quad \omega_k^{(n)} = \rho_k^{(t)} - i\sigma_k^{(t)}.$$

In the n-dimensional signature space $R_{n,t}$, corresponding to the space K_n (1), to the lattice under consideration there will correspond a lattice $[\omega]$, also repeated by multiplication, in every point of which all coordinates are real. To each basis point ω_k there will correspond the point with real coordinates

$$\omega_k^{(1)},\ \omega_k^{(2)},\ \dots,\ \omega_k^{(r)},\ \rho_k^{(1)},\ \sigma_k^{(1)},\ \dots,\ \rho_k^{(t)},\ \sigma_k^{(t)},$$

which will be a basis point of the lattice $[\omega]$. Every point of the lattice $[\omega]$ has coordinates equal to the values of n forms

$$\xi^{(i)} = \omega_1^{(i)} x_1 + \omega_2^{(i)} x_2 + \dots + \omega_n^{(i)} x_n, \qquad i = 1, 2, \dots, r,$$

$$\eta^{(j)} = \rho_1^{(j)} x_1 + \dots + \rho_n^{(j)} x_n, \quad \xi^{(j)} = \sigma_1^{(j)} x_1 + \dots + \sigma_n^{(j)} x_n, \qquad j = 1, 2, \dots, t,$$

for definite integral rational values $x_1, x_2, \dots, x_n$. The numbers

$$|\xi^{(1)}|,\ |\xi^{(2)}|, \dots,\ |\xi^{(r)}|, \quad \eta^{(1)2} + \zeta^{(1)2} = \tau^{(1)},\ \dots,\ \eta^{(t)2} + \zeta^{(t)2} = \tau^{(t)}$$

are the parameters of the point

$$(\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(r)}, \eta^{(1)}, \zeta^{(1)}, \dots, \eta^{(t)}, \zeta^{(t)}).$$

The region of the space $R_{n,t}$ in every point of which all parameters, with the exception of some k-th one, are not greater than unity, while the k-th may be arbitrary, we shall call the region $(1)^{(k)}$. Then the point 1 of the lattice $[\omega]$ lies on the boundary of the region $(1)^{(k)}$. We shall increase the k-th parameter of the point 1, leaving all its other parameters unchanged; then the volume of the corresponding norm body will increase, and when it exceeds by 2^n times the volume of the fundamental parallelepiped of the lattice, then, according to Minkowski' s theorem on a convex body with center of symmetry at a lattice point, inside the body there will be at least two lattice points lying in the region $(1)^{(k)}$, symmetric to one another with respect to the origin and therefore having correspondingly equal parameters. As the norm body is increased further, ever new pairs of lattice points will enter it. From each pair we shall choose only one point.

Arranging the points thus chosen in the order of increase of their k-th parameters, we obtain a sequence of points, which we shall call the **sequence** $\{1\}^{(k)}$.

§ 2. **Computation of the sequences $\{1\}^{(k)}$.** Consider the case when $r = n,\ t = 0$; the forms (1) will be

$$\xi^{(i)} = \omega_1^{(i)} x_1 + \omega_2^{(i)} x_2 + \dots + \omega_n^{(i)} x_n, \qquad i = 1, 2, \dots, n;$$

the coefficients of all forms are real numbers. The coordinates of every point of the lattice $[\omega]$ are conjugate integral algebraic numbers of the n-th order.

Let $k = 1$. In the sequence $\{1\}^{(1)}$ we shall agree to include points with positive coordinates $\xi^{(1)}$. Through the base points $\omega_2, \omega_3, \dots, \omega_n$ and the origin draw the $(n-1)$-dimensional plane Q_0. In it there will be an $(n-1)$-dimensional parallelepipedal system of points of the lattice $[\omega]$ with fundamental vectors $\overrightarrow{O\omega_2}, \overrightarrow{O\omega_3}, \dots, \overrightarrow{O\omega_n}$. We shall regard the whole lattice $[\omega]$ as consisting of an infinite set of such $(n-1)$-dimensional parallelepipedal systems, situated in parallel $(n-1)$-dimensional planes $\dots, Q_{-2}, Q_{-1}, Q_0, Q_1, Q_2, \dots$, passing respectively through the points $\dots, -2\omega_1, -\omega_1, 0, \omega_1, 2\omega_1, \dots$ and obtained from any one system, for instance the system of the plane Q_0, by translating it parallel to the vector $\overrightarrow{O\omega_1}$ onto the other planes.

Owing to the symmetry of the lattice $[\omega]$ with respect to the origin, it proves sufficient to consider only the planes $Q_0, Q_1, Q_2, \dots$. To compute the sequence $\{1\}^{(1)}$ one must successively find the points of the lattice $[\omega]$ lying on the planes $Q_0, Q_1, Q_2, \dots$ inside the region $(1)^{(k)}$. Denote by $p_1, p_2, \dots, p_n$ the values of the variables $x_1, x_2, \dots, x_n$ corresponding to some point of the sequence $\{1\}^{(1)}$; p_1 is the number of the plane Q_{p_1} on which the point lies. The method for computing $p_1, p_2, \dots, p_n$ is as follows: knowing p_1, we find the values p_2 from the conditions

$$-\alpha_1 < \begin{vmatrix} \omega_1^{(2)} & \omega_1^{(3)} & \cdots & \omega_1^{(n)} \\ \omega_3^{(2)} & \omega_3^{(3)} & \cdots & \omega_3^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ \omega_n^{(2)} & \omega_n^{(3)} & \cdots & \omega_n^{(n)} \end{vmatrix} p_1 + \begin{vmatrix} \omega_2^{(2)} & \omega_2^{(3)} & \cdots & \omega_2^{(n)} \\ \omega_3^{(2)} & \omega_3^{(3)} & \cdots & \omega_3^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ \omega_n^{(2)} & \omega_n^{(3)} & \cdots & \omega_n^{(n)} \end{vmatrix} p_2 < \alpha_1,$$

where $\alpha_1 = |A_{11}| + |A_{12}| + \cdots + |A_{1,n-1}|$, and $A_{11}, A_{12}, \dots, A_{1,n-1}$ are the minors of the first row of one of the determinants occurring in the condition; for each pair of values p_1, p_2, the values p_3 are found from the conditions

$$-\alpha_2 < \begin{vmatrix} \omega_1^{(3)} & \omega_1^{(4)} & \cdots & \omega_1^{(n)} \\ \omega_4^{(3)} & \omega_4^{(4)} & \cdots & \omega_4^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ \omega_n^{(3)} & \omega_n^{(4)} & \cdots & \omega_n^{(n)} \end{vmatrix} p_1 + \begin{vmatrix} \omega_2^{(3)} & \omega_2^{(4)} & \cdots & \omega_2^{(n)} \\ \omega_4^{(3)} & \omega_4^{(4)} & \cdots & \omega_4^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ \omega_n^{(3)} & \omega_n^{(4)} & \cdots & \omega_n^{(n)} \end{vmatrix} p_2 + \begin{vmatrix} \omega_3^{(3)} & \omega_3^{(4)} & \cdots & \omega_3^{(n)} \\ \omega_4^{(3)} & \omega_4^{(4)} & \cdots & \omega_4^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ \omega_n^{(3)} & \omega_n^{(4)} & \cdots & \omega_n^{(n)} \end{vmatrix} p_3 < \alpha_2,$$

where $\alpha_2 = |A_{21}| + |A_{22}| + \cdots + |A_{2,n-2}|$, and $A_{21}, A_{22}, A_{2,n-2}$ are the minors of the first row of one of the determinants of the conditions, and so on. Finally, for each computed system of values $p_1, p_2, \dots, p_{n-1}$, the values p_n are found from the conditions

$$-1 < \omega_1^{(1)} p_1 + \cdots + \omega_n^{(1)} p_n < 1, \quad -1 < \omega_1^{(2)} p_1 + \cdots + \omega_n^{(2)} p_n < 1, \dots$$

$$\dots, \quad -1 < \omega_1^{(n)} p_1 + \cdots + \omega_n^{(n)} p_n < 1.$$

On some of the first planes Q_i, inside the region $(1)^{(1)}$, there may occur points with negative coordinates $\xi^{(1)}$. For such points it is necessary to compute the points symmetric to them with respect to the origin and to include the latter in the sequence $\{1\}^{(1)}$.

If it is necessary to know whether in the sequence $\{1\}^{(1)}$ there is a point with a prescribed coordinate $\xi^{(1)} = h$, then the number i of the plane Q_i on which such a point may lie is restricted by the conditions $[\delta_1/|\Delta|] \le i \le [\delta/|\Delta|]$, where Δ is the determinant of the system (1), $\delta = |A_1|h + |A_2| + \cdots + |A_n|$, $\delta_1 = |A_1|h - |A_2| - \cdots - |A_n|$, and $A_1, A_2, \dots, A_n$ are the minors of the first column of the determinant Δ. The method described for computing the sequence $\{1\}^{(k)}$ for the case $r = n,\ t = 0,\ k = 1$ is easily extended also to all other cases that may occur.

§ 3. **Solution of the problem of the identity of two algebraic fields of the n-th degree.** Let ρ and ρ_1 be, respectively, some roots of two irreducible algebraic equations $f(x) = 0$, $f_1(x) = 0$ of degree n with integral rational coefficients and leading coefficients equal to one. Let Ω_ρ, Ω_{ρ_1} be the fields generated, respectively, by the roots ρ, ρ_1. If the fields $\Omega_\rho, \Omega_{\rho_1}$ are identical, then the discriminants D_ρ, D_{ρ_1} of the equations can differ only by rational square factors, i.e. one may put $D_\rho = \bar{\alpha}^2 D,\ D_{\rho_1} = \bar{\alpha}_1^2 D$, where $\bar{\alpha}, \bar{\alpha}_1, D$ are integral rational numbers and D no longer has square divisors.

Suppose that $[\omega]$, $[\tau]$ are the lattices corresponding to the rings of all integers of the fields $\Omega_\rho, \Omega_{\rho_1}$. In the case of identity of the fields, under a certain correspondence between the roots of the equations, the lattices will coincide; therefore, if we construct two sequences $\{1\}^{(k)}$—one in the lattice $[\omega]$, and the other in the lattice $[\tau]$—then, for equal k, these sequences must consist of the same points.

If in the first sequence we find the first n non-coplanar with the origin points $\alpha_i = a_{i1}\omega_1 + a_{i2}\omega_2 + \dots + a_{in}\omega_n$ $(i = 1, 2, \dots, n)$, then they must coincide, respectively, with the first n non-coplanar with the origin points $\beta_i = b_{i1}\omega_1 + b_{i2}\omega_2 + \dots + b_{in}\omega_n$ $(i = 1, 2, \dots, n)$ of the second sequence. The points $\alpha_i = a_{i1}\omega_1 + a_{i2}\omega_2 + \dots + a_{in}\omega_n$ $(i = 1, 2, \dots, n)$ are non-coplanar with the origin if the determinant of order n, $|a_{ik}|$, formed from the coefficients of the ω_i in the points, is not equal to zero.

Equating the corresponding points, we shall obtain n equations with integral rational coefficients. Since $\omega_1, \omega_2, \dots, \omega_n$ are expressed linearly in terms of $1, \rho, \dots, \rho^{n-1}$, and $\tau_1, \tau_2, \dots, \tau_n$ in terms of $1, \rho_1, \dots, \rho_1^{n-1}$, by means of these equations we can express ρ_1 in terms of $1, \rho, \dots, \rho^{n-1}$ and, conversely, ρ in terms of $1, \rho_1, \dots, \rho_1^{n-1}$, and hence find the transition functions from the equation $f(x) = 0$ to the equation $f_1(x) = 0$, and from the equation $f_1(x) = 0$ to the equation $f(x) = 0$. Since we cannot know in advance the necessary correspondence between the roots of the equations, in practice one must proceed as follows: having computed, in some sequence $\{1\}^{(k)}$ of the lattice $[\tau]$ (or $[\omega]$), the first n points $\alpha_1, \alpha_2, \dots, \alpha_n$ non-coplanar with the origin, we shall then compute the sequences $\{1\}^{(k)}$ in the other lattice $[\omega]$ (or $[\tau]$), taking $k = 1, 2, \dots, r + t$. In the case of identity of the fields $\Omega_\rho, \Omega_{\rho_1}$, in one of these sequences there will be found the first n points $\beta_1, \beta_2, \dots, \beta_n$, non-coplanar with the origin, such that for every point β_i the k-th coordinate will be equal to the k-th coordinate of the corresponding point α_i, while the remaining coordinates will be equal to the coordinates of the point α_i, perhaps not respectively, but taken in some other order.

One can decide the question of the identity of the fields $\Omega_\rho, \Omega_{\rho_1}$ without knowing a basis of the ring of all integers of either field. The basis of all integers of the field Ω_ρ, as is known, can be represented in the form

$$\omega_1 = 1, \quad \omega_2 = \frac{\rho + a_{22}}{\alpha_2}, \dots, \omega_n = \frac{\rho^{n-1} + a_{n2}\rho^{n-2} + \dots + a_{nn}}{\alpha_n},$$

where $0 \le a_{ij} \le \alpha_i - 1$; consequently, we shall have

$$\rho_1^k = x_{k1} + \frac{\rho + a_{22}}{\alpha_2} x_{k2} + \dots + \frac{\rho^{n-1} + a_{n2}\rho^{n-2} + \dots + a_{nn}}{\alpha_n} x_{kn}, \quad k = 1, \dots, n-1,$$

and hence

$$\alpha\rho_1^k = y_{k1} + \rho y_{k2} + \dots + \rho^{n-1} y_{kn}, \quad k = 1, 2, \dots, n-1,$$

where y_{ki} are integral rational numbers, and α is the common least multiple of the numbers $\alpha_1, \dots, \alpha_n$; α is found among the integral rational divisors of the number $\bar{\alpha}$. Thus the numbers $\alpha\rho_1, \alpha\rho_1^2, \dots, \alpha\rho_1^{n-1}$ belong to the ring $[1, \rho, \dots, \rho^{n-1}]$, and therefore all points of the lattice $[1, \alpha\rho_1, \dots, \alpha\rho_1^{n-1}]$ belong to the lattice $[1, \rho, \dots, \rho^{n-1}]$.

If in some sequence $\{1\}^{(k)}$ of the lattice $[1, \alpha\rho_1, \dots, \alpha\rho_1^{n-1}]$ one computes the first n non-coplanar points with origin $\alpha_1, \alpha_2, \dots, \alpha_n$, then in each sequence $\{1\}^{(k)}$ of the lattice $[1, \rho, \dots, \rho^{n-1}]$ we must now look for points coinciding with the points $\alpha_1, \alpha_2, \dots, \alpha_n$, not among the first n non-coplanar points of the origin sequences, but, for each point α_i, look separately for a point β_i whose k-th coordinate would be equal to the k-th coordinate of the point α_i, and whose remaining coordinates would be equal to the remaining coordinates of the point α_i, taken, perhaps, in a different order.

Example. Determine whether the fields $\Omega_\rho, \Omega_{\rho_1}$ are identical, where ρ, ρ_1 are any roots, respectively, of the equations:

$$x^4 - 2x^3 - 7x^2 + 8x + 1 = 0, \qquad x^4 - 9x^2 + 9 = 0.$$

The fields are purely real. The bases of the fields are

$$\left[\frac{\rho^3 + 2\rho^2 + \rho - 2}{7}, \ \rho^2, \ \rho, \ 1\right], \qquad \left[\frac{\rho^3}{3}, \ \frac{\rho^2}{3}, \ \rho, \ 1\right].$$

The discriminants of the fields are equal to 3600.

Having computed the roots of the equations, we form two systems of forms:

$$\xi^{(1)} = 8.77063x_1 + 11.22306x_2 + 3.35008x_3 + x_4,$$

$$\xi^{(2)} = -0.89765x_1 + 5.52289x_2 - 2.35008x_3 + x_4,$$

$$\xi^{(3)} = 0.42551x_1 + 1.24103x_2 + 1.11401x_3 + x_4,$$

$$\xi^{(4)} = -0.29849x_1 + 0.01299x_2 - 0.11401x_3 + x_4;$$

$$\xi_1^{(1)} = 7.33742x_1 + 2.61803x_2 + 2.80251x_3 + x_4,$$

$$\xi_1^{(2)} = -7.33742x_1 + 2.61803x_2 - 2.80251x_3 + x_4,$$

$$\xi_1^{(3)} = 0.40882x_1 + 0.38196x_2 + 1.07046x_3 + x_4,$$

$$\xi_1^{(4)} = -0.40882x_1 + 0.38196x_2 - 1.07046x_3 + x_4.$$

The points $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are obtained for the following quadruples of values x_1, x_2, x_3, x_4: $1, 0, 0, 0$; $2, 0, -1, 0$; $5, -1, -3, 2$; $10, -1, -5, 2$, and the points $\beta_1, \beta_2, \beta_3, \beta_4$ for the values: $1, 2, -1, 1$; $1, 3, 0, -1$; $2, -6, -1, 3$; $5, 14, -2, 5$.

The points α_1, β_1 will be

$$(8.7706,\ -0.8976,\ 0.4255,\ -0.2984), \quad (8.7709,\ -0.2988,\ -0.8976,\ 0.4255).$$

The first coordinates of the points are equal to an accuracy of four decimal places, while the remaining coordinates are equal, respectively. The analogous situation is observed for every other pair of corresponding points. Therefore, the fields are identical.

Equating the corresponding points, we obtain the system of equations:

$$\frac{1}{7}(\rho^3 + 2\rho^2 + \rho - 2) = \frac{1}{3}(\rho_1^3 + 2\rho_1^2 - 3\rho_1 - 3),$$

$$\frac{1}{7}(2\rho^3 + 4\rho^2 - 5\rho - 4) = \frac{1}{3}(\rho_1^3 + 3\rho_1^2 - 3),$$

$$\frac{1}{7}(5\rho^3 - 3\rho^2 - 16\rho + 4) = \frac{1}{3}(2\rho_1^3 + 6\rho_1^2 - 3\rho_1 - 9),$$

$$\frac{1}{7}(10\rho^3 + 13\rho^2 - 25\rho + 1) = \frac{1}{3}(5\rho_1^3 + 14\rho_1^2 - 6\rho_1 - 15),$$

from which we find the transition functions:

$$\rho = \frac{\rho_1^3 + \rho_1^2 - 6\rho_1 - 3}{3}, \qquad \rho_1 = \frac{-\rho^3 + 3\rho^2 - 3}{7}.$$

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CITED LITERATURE

1. B. N. Delone, D. K. Faddeev, *Theory of Irrationalities of the 3rd Degree*, M.–L., 1940.

Note: Figure translations are in progress. See original paper for figures.