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Abstract

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MATHEMATICS

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ON THE RATE OF CONVERGENCE OF THE METHOD OF ORTHOGONAL PROJECTIONS FOR THE FIRST BOUNDARY-VALUE PROBLEM FOR EQUATIONS OF POLYHARMONIC TYPE

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§ 1. Consider the elliptic equation

$$\sum_{k=0}^{2m} A_k \frac{\partial^{2m} u(x,y)}{\partial x^{2m-k} \partial y^k} = A_{2m} f(x,y). \qquad (m = 1, 2, \dots) \tag{1}$$

with constant real coefficients A_k. Let us split the set of roots of the corresponding characteristic equation

$$\sum_{k=0}^{2m} A_k \mu^k = 0 \tag{2}$$

into two complex-conjugate sets $\{\mu_k\}$, $\{\overline{\mu}_k\}$ $(k = 1, 2, \dots, m)$, and introduce the complex-conjugate operators M_m, $\overline{M}_m$, where

$$M_m = \prod_{k=1}^{m} B_k, \qquad B_k = \frac{\partial}{\partial y} - \mu_k \frac{\partial}{\partial x}.$$

Then equation (1) can be represented in the form

$$M_m \overline{M}_m u = f.$$

Let the domain S be bounded by a piecewise-smooth closed Jordan curve Γ. We shall assume that in $\overline{S}$, $f \in \operatorname{Lip} \alpha$. In the domain S one seeks a solution of equation (1) satisfying on the contour Γ the boundary conditions

$$u\big|_\Gamma = d^k u / dn^k\big|_\Gamma = 0 \qquad (k = 1, 2, \dots, m-1). \tag{3}$$

By a solution of the problem we shall mean a function $u(x,y)$ satisfying, in addition to (1) and (3), the following requirements: $u \in C_{m-1}(\overline{S})$, $u \in C_{2m}(S)$, $\overline{M}_m u \in L_2(S)$.

§ 2. Let us note several facts essential for the proof of the main results.

We shall assign a complex function $\varphi(x,y)$ to the class $Q_{M_m}(S)$ if it has the following properties: $\varphi \in C_m(S)$, $\varphi \in L_2(S)$, and in the domain S, $M_m\varphi = 0$.

Let $\varphi \in Q_{M_m}(S)$, and let (x,y) be an arbitrary interior point of the domain S. Let δ be the distance from this point to the nearest point of the contour Γ, $\delta_1 = \min\{1/2, \delta\}$, and $D(\delta_1)$ the circle of radius δ_1 with center at the point (x,y).

Green's formula for the operator M_m makes it possible to establish the estimates

$$\left|\frac{\partial^\sigma \varphi(x,y)}{\partial x^{\sigma_1}\partial y^{\sigma_2}}\right| \le N_{\sigma,m}\frac{|\log\delta_1|}{\delta_1^{\sigma+1}}\left[\iint_{D(\delta_1)}|\varphi|^2\,dx\,dy\right]^{1/2} \qquad (\sigma = 0,1,\ldots). \quad (4)$$

Here $N_{\sigma,m}$ is a constant independent of the choice of φ and of the point (x,y).

Let among the numbers of the set $\{\mu_k\}$ there be q distinct numbers $(\mu_1,\mu_2,\ldots,\mu_q)$, let the multiplicity of the number μ_l be equal to β_l $(l = 1,2,\ldots,q)$, and let $\beta = \max\{\beta_l\}$. If $\varphi \in Q_{M_m}(S)$, then in S the representation

$$\varphi(x,y) = \sum_{l=1}^{q}\sum_{r=0}^{\beta_l-1} y^r \omega_{l,r}(z_l), \quad (5)$$

holds, where $z_l = x + \mu_l y$; $\omega_{l,r}(z_l)$ are certain functions, analytic (each in its own argument z_l) at all points $(x,y) \in S$.

Consider the system of functions

$$\psi_k(x,y) = \begin{cases} y^k, & \text{if } k = 0,1,\ldots,\beta-1, \\ y^r z_l^\sigma, & \text{if } k = \beta + m(\sigma-1) + \sum_{j=1}^{l-1}\beta_j + r, \end{cases}$$

where the indices σ, l, r run through the values: $\sigma = 1,2,\ldots;$ $l = 1,2,\ldots,q;$ $r = 0,1,\ldots,\beta_l - 1$. It is obvious that $\psi_k \in Q_{M_m}(S)$ $(k = 0,1,\ldots)$. Denote by $\{p_k(x,y)\}$ the system of polynomials obtained as a result of orthonormalizing the system $\{\psi_k(x,y)\}$ over the domain S. Relying on formula (5), on a theorem proved by A. I. Markushevich (^{1}, p. 428), and on inequality (4), one can show that theorem 1 holds.

Theorem 1*. The system $\{p_k(x,y)\}$ is closed in the class $Q_{M_m}(S)$, and the Fourier series of a function $\varphi \in Q_{M_m}(S)$ converges to φ uniformly in every closed domain contained entirely inside S.

§ 3. Let $g(x,y)$ be an arbitrary solution of the equation $M_m g(x,y) = f(x,y)$, satisfying the requirements: $g \in C_m(S)$, $g \in L_2(S)$. Then theorem 2 holds.

Theorem 2. If $u(x,y)$ is the solution of the problem formulated in § 1, then the equality

$$u(x,y) = \frac{1}{2\pi i}\iint_S \left[g(\xi,\eta) - \sum_{k=0}^{\infty} g_k p_k(\xi,\eta) \right] G_{\overline{M}m}(\xi - x, \eta - y)\, d\xi\, d\eta, \qquad (6)$$

holds, where $G_{\overline{M}m}(x,y)$ is the principal singular solution of the equation $\overline{M}_m G_{\overline{M}m}(x,y) = 0$, and g_k are the Fourier coefficients of the function $g(x,y)$.

The proof of theorem 2 is based on theorem 1, as well as on Green' s formula for the operator $\overline{M}_m$. In the particular case when $\overline{M} = \Delta$, an analogous theorem was proved by Z. E. Rafalson (^{2}). For $m = 1$ and $m = 2$ the corresponding proofs are given in (^{3,4}).

§ 4. As the n-th approximation to the solution of the problem $u(x,y)$, we shall consider the function

$$u_n(x,y) = \frac{1}{3\pi i}\iint_S \left[g(\xi,\eta) - \sum_{k=0}^{n-1} g_k p_k(\xi,\eta) \right] G_{\overline{M}m}(\xi - x, \eta - y)\, d\xi\, d\eta. \qquad (7)$$

Lemma. The estimates hold

$$\left| \frac{\partial^\sigma (u - u_n)}{\partial x^{\sigma_1} \partial y^{\sigma_2}} \right|_{(x,y)} \le \widetilde{N}_{\sigma,m} h_\sigma(\delta_1) \left[\sum_{k,n}^{\infty} |g_k|^2 \right]^{1/2} \qquad (\sigma = 0, 1, \ldots),$$

where $\widetilde{N}_{\sigma,m}$ does not depend on n or on the choice of the point $(x,y) \in S$, $h_\sigma(\delta_1) = 1$ for $\sigma = 0, 1, \ldots, m-2$; $h_\sigma(\delta_1) = |\log \delta_1|$ for $\sigma = m-1$, $h_\sigma(\delta_1) = \delta_1^{-\sigma-1+m} |\log \delta_1|^2$ for $\sigma \ge m$.

Proof. Form the difference of the functions (6) and (7). It may be differentiated under the integral sign $m-1$ times. To prove the lemma for $\sigma = 0, 1, \ldots, m-2$, it is enough to apply Bunyakovsky' s inequality to the derivatives of the indicated difference. Let now $\sigma = m-1$. Then the domain S of the corresponding integral should be divided into two parts: $D(\delta_1/2)$ and $S - D(\delta_1/2)$. The integral over

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* In proving this theorem in the case $m \ge 2$, it was necessary to assume that the domain S is star-shaped.

the domain $S - D(\delta_1/2)$ can be estimated by Bunyakovsky' s inequality, since there the function $G_{\overline{M}_m}$ has no singular points. In estimating the integral over the domain $D(\delta_1/2)$, one must take into account that in it the derivatives of $G_{\overline{M}_m}$ of order $m{-}1$ have a singularity of type $1/r$, and, by virtue of the estimates (4),

$$\max_{D(\delta_1/2)} \left| \sum_{k=n}^{\infty} g_k p_k'(x,y) \right| \leq N_{0,m} (\delta_1/2)^{-1} |\log(\delta_1/2)| \left[\sum_{k=n}^{\infty} |g_k|^2 \right]^{1/2} .$$

Finally, let $\sigma \geq m$. It is clear that $\partial^{m-1}(u - u_n)/\partial x^{\tau_1} \partial y^{\tau_2} \in Q_{T_{2m}}(S)$, where $T_{2m} = M_m \overline{M}_m$. Consequently, for this function one may write an inequality of type (4). Estimating in the indicated inequality the integral (with the aid of the already obtained estimate for $\partial^{m-1}(u - u_n)/\partial x^{\tau_1} \partial y^{\tau_2}$), we prove what is required.

§ 5. The rate of convergence of the method of orthogonal projections is in direct dependence on the degree of smoothness of the solution $u(x, y)$. Namely, the following theorem holds.

Theorem 3. Let the domain S be bounded by a Jordan curve Γ, whose curvature, as a function of arc length, belongs to the class $\operatorname{Lip} \alpha$. Let in $\overline{S}$ $u \in \operatorname{Lip}(k, \lambda)$, $g \in \operatorname{Lip}(k - m, \lambda)$, where $k \geq 2m - 1,\ 0 < \lambda \leq 1$. Then

$$\left| \frac{\partial^{\sigma}(u - u_{\beta+mn})}{\partial x^{\sigma_1} \partial y^{\sigma_2}} \right|_{(x,y)} \leq C_{\sigma,m} h_{\sigma}(\delta_1) \left[\frac{a(n)}{n} \right]^{k-m-(\beta-1)+\lambda} ,$$

where $a(n) = \log n$ if the contour Γ is not analytic, and $a(n) = 1$ if Γ is an analytic contour. Here $\sigma = 0, 1, ...$; $C_{\sigma,m}$ is a constant not depending on n or on the choice of the point $(x, y) \in S$.

Proof. Consider the function $\varphi = g{-}\overline{M}u$. It is clear that in $\overline{S}$, $\varphi \in \operatorname{Lip}(k{-}m, \lambda)$. On the other hand, by Theorem 2,

$$\varphi = \sum_{\tau=0}^{\infty} g_\tau p_\tau,$$

and therefore this function belongs to the class $Q_{M_m}(S)$ and can be represented in the form (5). From formula (5) and the inclusion $\varphi \in \operatorname{Lip}(k - m, \lambda)$ it follows that in $\overline{S}$, $\omega_{l,r}(z_l) \in \operatorname{Lip}(k - m - \beta_l + 1, \lambda)$. Therefore, relying on the results established by Curtiss (6) and Sewell (7), one may assert that for each n there exist polynomials $P_{l,r;n}(z_l)$ of degree n in z_l such that, uniformly in the domain $\overline{S}$,

$$\left|\varphi(x,y) - \sum_{l=1}^{q} \sum_{r=0}^{\beta_l - 1} y^r P_{l,r;n}(z_l)\right| \leq C \left[\frac{a(n)}{n}\right]^{k-m-(\beta-1)+\lambda}.$$

Square this inequality and integrate over the domain S. Then replace the polynomial standing on the left-hand side by a segment of the Fourier series of the function φ, and the function φ by its Fourier series. We obtain

$$\sum_{\tau=\beta+mn}^{\infty} |g_\tau|^2 \leqslant C^2 \,\mathrm{Pl.}\, S \left[\frac{a(n)}{n}\right]^{2[k-m-(\beta-1)+\lambda]},$$

which, in combination with the lemma, proves the theorem*.

§ 6. The problem under consideration admits a variational formulation.

Indeed, the quantity

$$\overline{M}_m u_n = g - \sum_{k=0}^{n-1} g_k p_k$$

can be found from the condition that the integral

$$\iint_S |\overline{M}_m u_n|^2 \, dx \, dy$$

be minimized, if as admissible functions one takes functions of the form

$$\overline{M}_m u_n(x,y) = g(x,y) - \sum_{k=0}^{n-1} c_k \psi_k(x,y). \tag{8}$$

Let us note (for $m = 2$, see (5)) that, in order to determine the partial derivatives $\partial^m u_n / \partial x^{\sigma_1} \partial y^{\sigma_2}$, it is sufficient to solve a system consisting of $m+1$ linear algebraic equations of type (8), corresponding to different ways of splitting the roots of equation (2) into two complex-conjugate sets. The derivatives of lower order (including the function $u_n(x,y)$ itself) can then be found with the aid of curvilinear integrals, computed elementarily, since their integrands are polynomials in x and y. Thus, in order to obtain the n-th approximation it is in fact not necessary to compute the singular integral (7).

§ 7. Let us compare the rate of convergence of the method under investigation with that of Ritz' s method. We restrict ourselves to the case $m = 1$. From Theorem 3 we conclude that the error $|u - u_{n+1}|$ of the method of orthogonal projections is proportional to $(\log n/n)^{k-1+\lambda}$. Under the same assumptions on the degree of smoothness of the exact solution, the error $|u - u_{(n+1)^2}|$ of Ritz' s method is proportional to $\sqrt{\log n}/n^{k-1+\lambda}$ (([9], p. 384)**. Thus, a practically identical error estimate corresponds, in Ritz' s method, to $(n+1)^2$ parameters, and in the method of orthogonal projections to $n+1$. With an equal number of parameters (i.e., with approximately the same amount of computation), the error estimate in the method under investigation will be considerably better than the estimate for Ritz' s method. Finally, let us note that, in contrast to Ritz' s method, in the method of orthogonal projections convergence is ensured (in any case at interior points of the domain S) for derivatives of arbitrary order.

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REFERENCES

[1] A. I. Markushevich, *Theory of Analytic Functions*, 1950.
[2] E. F. Rafalson, DAN, 64, No. 6, 799 (1949).
[3] M. I. Kliot-Dashinskii, Scientific Notes of Leningrad State University, 146, ser. phys. sciences, issue 8, 131 (1952).
[4] M. I. Kliot-Dashinskii, Collection of scientific works of LISI, issue 17, 11 (1954).
[5] M. I. Kliot-Dashinskii, Uspekhi Mat. Nauk, 11, No. 6, 247 (1956).
[6] J. H. Curtiss, Bull. Am. Math. Soc., 42, 873 (1936).
[7] W. E. Sewell, Proc. Nat. Acad. Sci. USA, 23, No. 9, 491 (1937).
[8] S. N. Mergelyan, Uspekhi Mat. Nauk, 7, No. 2, 31 (1952).
[9] I. Yu. Kharrik, Matematicheskii sbornik, 37, No. 2, 353 (1955).

* Relying on the results established by S. N. Mergelyan (([8]), p. 109), one may also consider the case when Γ is a piecewise-smooth Jordan curve.

** In both methods we number the approximations by the number of parameters determined from the minimum condition for the corresponding integrals. In Ritz' s method the approximation $u_{(n+1)^2}$ is sought in the form

$$\Omega(x,y)\sum_{\sigma,\tau=0}^{n} c_{\sigma,\tau}x^{\sigma}y^{\tau}.$$

Note: Figure translations are in progress. See original paper for figures.

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