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Abstract

Full Text

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ON A CERTAIN CLASS OF SINGULAR INTEGRAL EQUATIONS

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§ 1. Consider the integral equation

$$u(y,t)-\lambda\int_0^t d\tau\int_{-\infty}^{+\infty}K((y-\eta)^2,t-\tau)u(\eta,\tau)\,d\eta=f(y,t), \qquad (1)$$

where

$$K((y-\eta)^2,t-\tau)=$$

$$=\frac{1}{(t-\tau)^{3/2}}\int_0^{\infty}\rho(z)\left[1-\frac{(y-\eta)^2}{2a^2(z)(t-\tau)}\right]\exp\left[-\frac{(y-\eta)^2}{4a^2(z)(t-\tau)}\right]dz, \qquad (2)$$

$$\rho(z)=(z^2+a_1^2)^{-3/2}(z^2+a_1^2)^{-1/2}, \qquad a^2(z)=a_2^2(z^2+a_1^2)/(z^2+a_2^2), \qquad (3)$$

$f(y,t)$ is a given function in the domain $t>0,\ -\infty<y<+\infty$, and $u(y,t)$ is the unknown function. Such a singular integral equation occurs in solving the heat-exchange problem for a system of bodies that are in thermal contact with one another.

§ 2. For the solution of equation (1), consider the Fourier transform of generalized functions defined by linear continuous functionals of the form

$$(T,\varphi)=\int_{-\infty}^{+\infty}T(x)\varphi(x)\,dx. \qquad (4)$$

We denote by $T(\Phi)$ the totality of all generalized functions acting in the space $\Phi(s,k,k_p,z^p,z_p^p)$ (^{1}). The Fourier transform of the function $f(x)$ will be denoted briefly by $\tilde{f}(x)$.

Following Schwartz' s method (^{1}), as the basis for defining the Fourier transform for any generalized function we set the equality

$$(\tilde{T}, \tilde{\varphi}) = (T(x), \varphi(-x)). \tag{5}$$

The functional defined by this formula acts in the space $\tilde{\Phi}$, dual with respect to Φ.

If the Fourier transform acts in the space Φ of basic functions $\varphi(x)$, then on the basis of (5) we have

$$(T(x+\eta), \varphi(-x)) = (T(x), \varphi(-x+\eta)) =$$

$$= (\tilde{T}(s), e^{2\pi i\eta s}\tilde{\varphi}(s)) = (e^{2\pi i\eta s}\tilde{T}(s), \tilde{\varphi}(s)). \tag{6}$$

§ 3. Let us now consider the integral equation (1), which can be rewritten in the following form:

$$u(y,t) - \lambda \int_0^t d\tau \int_{-\infty}^{+\infty} K(\eta, t-\tau) u(\eta+y, \tau)\, d\eta = f(y,t). \tag{1'}$$

Multiplying (1′) by $\varphi(-y)$ and integrating from $-\infty$ to $+\infty$, we obtain

$$(u(y,t), \varphi(-y)) - \lambda \int_0^t d\tau \int_{-\infty}^{+\infty} K(\eta, t-\tau)(u(\eta+y,\tau)\varphi(-y))\, d\eta = (f(y,t), \varphi(-y)).$$

Applying formulas (5) and (6), we have:

$$(\tilde{u}(s,t), \tilde{\varphi}(s)) - \lambda \int_0^t d\tau \int_{-\infty}^{+\infty} K(\eta, t-\tau)(e^{2\pi i s\eta}\tilde{u}(s,\tau), \tilde{\varphi}(s))\, d\eta = (\tilde{f}(s,t), \tilde{\varphi}(s))$$

or

$$\left(\tilde{u}(s,t) - \lambda \int_0^t \left\{ \int_{-\infty}^{+\infty} K(\eta, t-\tau) e^{2\pi i s\eta}\, d\eta \right\} \tilde{u}(s,\tau)\, d\tau - \tilde{f}(s,t), \tilde{\varphi}(s) \right) = 0.$$

Thus, in the space $T(\tilde{\Phi})$ the following equation acts:

$$\tilde{u}(s,t) - \lambda \int_0^t K_0(s, t-\tau)\tilde{u}(s,\tau)\, d\tau = \tilde{f}(s,t), \tag{7}$$

where

$$K_0(s,t-\tau) = \int_{-\infty}^{+\infty} K(\eta, t-\tau) e^{2\pi i s\eta}\, d\eta =$$

$$= 16\pi^{3/2} s^2 \int_0^{\infty} \rho(z) a^3(z) e^{-4\pi^2 s^2 a^2(z)(t-\tau)}\, dz. \tag{8}$$

The problem of uniqueness of the solution of equation (1) and of equation (7) is equivalent by virtue of the isomorphism between the spaces $T(\Phi)$ and $T(\tilde{\Phi})$.

Let us consider equation (7) first from the classical point of view. Applying the operational method, we find the solution in closed form:

$$\tilde{u}(s,t) = \tilde{f}(s,t) + \lambda \int_0^t \Gamma(s,t-\tau) \tilde{f}(s,\tau)\, d\tau, \tag{9}$$

where

$$\Gamma(s,t-\tau) = \frac{8\pi^{7/2}(|\nu| - \nu)}{(\nu^2-1)(a_1^2\nu^2 - a_2^2)}\, s^2 \exp[-4\pi^2 d_0 s^2 (t-\tau)] +$$

$$+16 a_2^3 \pi^{5/2} (\nu+1)^2 \int_0^{\infty} \frac{z^2 s^2 \exp[-4\pi^2 a^2(z) s^2 (t-\tau)]}{(z^2+a_2^2)^2(\nu^2 z^2 + a_2^2)}\, dz\, *; \tag{10}$$

$$d_0 = (a_1^2\nu^2 - a_2^2)/(\nu^2-1), \qquad \nu = (a_1^2 - a_2^2)/2\pi^{3/2}\lambda - 1. \tag{11}$$

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- This formula has meaning if $\nu \neq \pm 1$, $\nu \neq \pm a_2/a_1$. If $\nu = 1;\ a_2/a_1;\ -a_2/a_1$, then the existing formulas can be obtained by a limiting passage. If, however, $\nu = -1$, then $\lambda = \infty$. Therefore the last case is excluded.

For the existence of the integral (9) we require that the function $t^\sigma \tilde{f}(s,t)$, for $0 \le \sigma < 1$, be bounded for fixed s.

It is easily verified that the resolvent (10) satisfies the equations

$$\Gamma(s,t-\tau) = K_0(s,t-\tau) + \lambda \int_\tau^t \Gamma(s,t-t_1) K_0(s,t_1-\tau)\, dt_1, \tag{12}$$

$$\Gamma(s,t-\tau) = K_0(s,t-\tau) + \lambda \int_\tau^t K_0(s,t-t_1) \Gamma(s,t_1-\tau)\, dt_1. \tag{13}$$

From (8) and (10) we see that $K_0(s, t-\tau)$ and $\Gamma(s, t-\tau)$ are entire functions with respect to s, and the order of their growth is equal to 2. Therefore the functions $K_0(s, t-\tau)$ and $\Gamma(s, t-\tau)$ are multipliers in the space Z_r^r for $r > 2$ [1].

If $\tilde{f}(s, t)$ is a generalized function acting in the space $Z_r^{r'}$ $(r > 2)$, then the generalized function $\tilde{u}(s, t)$, defined by formula (9), acts in $Z_r^{r'}$. Therefore, carrying out the usual computations with the aid of equalities (12) and (13), one can prove that $\tilde{u}(s, t)$ satisfies equation (7).

For the uniqueness theorem for the solution of equation (7) we can prove that the equation

$$\tilde{u}(s, t) - \lambda \int_0^t K_0(s, t-\tau) \tilde{u}(s, \tau)\, d\tau = 0, \tag{7'}$$

has only the zero solution acting in Z_r^r. If $r > 2$, then the class Φ of basic functions corresponding to the problem of existence and uniqueness of the solution is dual to the class Z_r^r, i.e. $Z_{r'}^{r'}$, where $1/r' + 1/r = 1$. As functionals on the space $Z_{r'}^{r'}$, in particular, all ordinary functions $f(x)$ satisfying the inequality

$$|f(x)| \leq c_1 e^{c|x|^{r'-\delta}}, \tag{14}$$

serve, where δ is an arbitrary positive number.

Let us note that the numbers r and r' can be taken arbitrarily close to the number 2; therefore the difference $r' - \delta$ can be denoted by $2 - \varepsilon$, where ε is an arbitrarily small positive number. Thus we arrive at the following theorem.

Theorem 1. If the function $f(y, t)$ satisfies the inequality

$$|t^\sigma f(y, t)| \leq c_1 e^{c|y|^{2-\varepsilon}}, \qquad \varepsilon > 0, \quad 0 \leq \sigma < 1, \tag{15}$$

then the solution of equation (1) exists in the class of generalized functions $u(y, t)$ for which, for every $t > 0$, the generalized function $t^\sigma u(y, t)$ belongs to the space $T(Z_{2-\delta}^{2-\delta})$, $\delta > 0$, and this solution is unique.

If for the equation under consideration $\nu \geq 0$ or $d_0 \geq 0$, i.e. $\lambda \leq a_1(a_1 + a_2)/2\pi^{1/2}$, then from (10) we see that

$$|\Gamma(s, t-\tau)| \leq c_1 |s|^2 e^{c|\mu|^2},$$

where μ is the imaginary part of s. In this case the integral equation (1)–(7) will be called **regular**.

For such an equation (1)–(7), instead of the space Z_r^r one can use the more elementary space Z_2, where the resolvent $\Gamma(s, t-\tau)$ and $K_0(s, t-\tau)$ are also multipliers. The space dual to it will be K_2. As functionals on K_2 all functions $f(x)$ satisfying the inequality

$$|f(x)| \leq c e^{c|x|^2}$$

are admitted.

Thus we arrive at the following theorem.

Theorem 2. If $f(y,t)$ satisfies the inequality

$$\left|t^{\delta} f(y,t)\right| \le c_1 e^{c|y|^2}, \tag{16}$$

then, for $\lambda \le a_1(a_1+a_2)/2\pi^{3/2}$, equation (1) has a unique solution in the class of generalized functions $u(y,t)$ for which the functions $t^{\delta}u(y,t)$ belong to $T(K_2)$.

We shall now prove that if

$$\lambda < a_1(a_1+a_2)/2\pi^{3/2}, \tag{17}$$

then, under additional conditions, besides (15) or (16), the solution of equation (1) exists in the class of ordinary functions.

For this purpose let us consider the integral equation (1) when $t^{\delta} f(y,t)$ belongs to s with respect to the first argument. In this case, applying the ordinary Fourier transform to equation (1), we obtain equation (7). The solution of this equation is expressed by formula (9). It is easy to see that under condition (17) $\nu \ge 0$ or $d_0 = b^2 > 0$. To find the solution of equation (1), we apply the inverse Fourier transform to (9). In doing so, to the product $\Gamma(s,t-\tau)F(s,\tau)$ we apply the convolution formula. Thus we finally obtain the solution of equation (1) in the form

$$u(y,t) = \lambda \int_0^t d\tau \int_{-\infty}^{+\infty} R(y-\eta, t-\tau) f(\eta,\tau)\, d\eta + f(y,t), \tag{18}$$

where

$$\begin{aligned} R(y-\eta,t-\tau) = {} & \frac{\pi}{2}\,\frac{(|\nu|-\nu)}{b^3(\nu-1)^2(t-\tau)^{3/2}}\left[1-\frac{(y-\eta)^2}{2b^2(t-\tau)}\right]\exp\left[-\frac{(y-\eta)^2}{4b^2(t-\tau)}\right] \\ & + \frac{(\nu+1)^2}{(t-\tau)^{3/2}}\int_0^{\infty}\frac{z^2}{(z^2+a_1^2)^{3/2}(z^2+a_2^2)^{1/2}(\nu^2 z^2+a_2^2)} \\ & \times\left[1-\frac{(y-\eta)^2}{2a^2(z)(t-\tau)}\right]\exp\left[-\frac{(y-\eta)^2}{4a^2(z)(t-\tau)}\right] dz. \end{aligned} \tag{19}$$

It is easy to verify that $R(y-\eta,t-\tau)$ satisfies the resolvent equations.

The class of functions $f(y,t)$ satisfying the inequalities

$$t^{\sigma}|f(y,t)| \le c_1 e^{c|y|^p}, \tag{20}$$

$$t^{\sigma}|f(y_1,t) - f(y_2,t)| \leq c_3 e^{c_2 \bar{y}^p} |y_1 - y_2|^{\alpha} \quad (0 < \alpha \leq 1),$$

where $\bar{y} = \max(|y_1|, |y_2|)$, will be denoted by H_p.

It is obvious that for any function $f(y,t) \in H_p$ with $p < 2$ the integral in (19) exists, while for $p = 2$ it exists for small values of t.

By direct verification one can establish that the function $u(y,t)$ defined by formula (18) satisfies the integral equation (1), if $f(y,t) \in H_p$ $(p \leq 2)$.

One can give an example where, for $\lambda \geq a_1(a_1 + a_2)/2\pi^{3/2}$, the solution of (1) is not an ordinary function, despite the fact that $f_t(y,t) \in H_p$ $(p \leq 2)$. All these results lead us to the following theorem.

Theorem 3. In order that the solution of equation (1), for $f(y,t) \in H_p$ $(p \leq 2)$, be an ordinary function, it is necessary and sufficient that λ satisfy inequality (17).

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References

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Note: Figure translations are in progress. See original paper for figures.