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Abstract

Full Text

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AN EXAMPLE OF A BIFURCATION THAT DOES NOT LEAD TO THE APPEARANCE OF UNSTABLE FORMS OF STATIONARY MOTION

The existence of one or several forms of equilibrium or stationary motion of a mechanical system is, as a rule, determined by a concrete value of some parameter that essentially determines the state of the system. Such a parameter, for example, in the case of compression of Euler' s column (see Fig. 1a), is the magnitude of the compressive force P, while in the case of a plane pendulum whose hinge axis rotates in a horizontal plane (see Fig. 1b), the essential parameter is the value of the angular velocity of rotation of the hinge about the vertical axis ω.

To a certain interval of parameters there may correspond a unique (principal) form of equilibrium or stationary motion. For example, to the rectilinear form of Euler' s column, as the unique form of equilibrium, there corresponds the value of the compressive force P lying in the interval

$$0 \leqslant P \leqslant P_e = \pi^2 EI/l^2. \tag{1}$$

In the case of the pendulum, a stationary form of motion in which the center of gravity of the pendulum remains motionless will occur if

$$0 \leqslant \omega^2 \leqslant \omega_{\text{cr}}^2 = mga/(A - C), \tag{2}$$

where A is the moment of inertia of the pendulum about the hinge axis; C is the moment of inertia with respect to the straight line perpendicular to the hinge axis and passing through the center of gravity (the line of the pendulum); m is the mass of the pendulum; a is the distance between the center of gravity and the hinge axis.

Fig. 1

Other intervals of the essential parameter may correspond, along with the principal form of equilibrium or stationary motion, also to other forms. For Euler' s column, when

Fig. 3

Figure 1: Fig. 3

$$P_e < P \leqslant 4P_e \tag{3}$$

there exist three forms of equilibrium: the rectilinear (principal) one and two curvilinear ones, mirror images of one another (Figs. 1a and 2a).

Fig. 2

In the case of the rotating pendulum, if

$$\omega > \omega_{\mathrm{cr}} = \sqrt{mga/(A-C)}, \tag{4}$$

then, along with the principal form (the center of gravity is motionless and occupies the lowest position), a stationary form of motion is possible in which

the center of gravity executes conical motion. In this case the line of the pendulum deviates from the vertical by an angle θ, determined by the relation*

$$\cos\theta = \frac{mga}{(A-C)\omega^2}. \tag{5}$$

Values of the parameter lying on the boundary of existence of one or several forms are called, as is well known, **bifurcation** values.

In a number of cases, in particular in the examples given above, the new equilibrium forms of stationary motion develop from the basic form in such a way that a small difference between the value of the essential parameter and its bifurcation value corresponds to a small deviation of the new forms from the initial form. It is as though the basic form branches into several forms of equilibrium or stationary motion (Figs. 2a and 2b).

Fig. 3

In all cases known to the author of the present article, the newly appearing forms (at least one of them) are stable, while the basic form of equilibrium or stationary motion ceases to be stable.

Below an example is presented in which the basic form, together with the newly appearing forms of stationary motion, remains stable for all values of the essential parameter. This example was discovered as a result of interesting experiments by S. V. Malashenko on the stability of rotation of an elongated axisymmetric rigid body suspended on a string (see Fig. 3a). At a certain, quite definite value of the angular velocity, the basic (vertical) form of stationary rotation of the body ceased to be stable. The stable form of stationary motion became the form shown in Fig. 3b, and the angle α, as was to be expected, increased as the angular velocity increased. However, when the value of the angular velocity exceeded its bifurcation value by a comparatively small amount, the basic form of stationary motion (the string vertical) again became stable.

When the experiments were carried out in vacuum the same thing was obtained.

However, with especially careful balancing of the suspended cylindrical body—in particular, when the body was suspended on a kapron thread and the speed of its rotation was changed slowly—it was possible to observe the basic form at all values of the angular velocity.

Later, carefully conducted experiments revealed an analogous phenomenon in the region of lower speeds. As was not difficult to predict, the new form of dynamic equilibrium turned out to be the form shown in Fig. 3c.

The foregoing shows that here there is a branching (bifurcation) of forms of dynamic equilibrium, but without loss of stability of the basic form.***

If the thread is regarded as absolutely flexible, and the resistance to rotation is —

* In addition, for any value of the angular velocity ω there always exists an unstable form of stationary motion of the pendulum, determined by the angle $\theta = \pi$. This form is isolated and will not be considered further.

** In other cases (well known in the theory of nonlinear oscillations) the new forms may differ sharply from the basic one, and a continuous transition of one form into another proves impossible (1).

*** This thereby casts serious doubt on the possibility of a purely hydrodynamic interpretation of the loss of stability of the basic form of rotation of a rigid body suspended on a string, when there is a cavity inside the body completely filled with liquid.

then the calculation of the bifurcation values of the angular velocity leads to the consideration of the equations

$$Ml^2 \sin\alpha \cos\alpha\, \omega^2 - Mal \cos\alpha \sin\varphi\, \omega^2 = Mgl \sin\alpha,$$

$$Ma^2\omega^2 \sin\varphi \cos\varphi - Mal\omega^2 \sin\alpha \cos\varphi + A\omega^2 \sin\varphi \cos\varphi -$$

$$-C\omega^2 \cos\varphi \sin\varphi = Mga \sin\varphi \qquad (6)$$

with respect to the angles α and φ, which determine the form of the stationary motion.

Fig. 4

Figure 2: Fig. 4

In order to find the bifurcation values of the angular velocity ω, in equations (6) one must put α and φ small and retain in them terms of the first order of smallness with respect to these quantities. We obtain

$$(Ml^2\omega^2 - Mgl)\alpha - Mal\omega^2\varphi = 0,$$

$$-Mal\omega^2\alpha + [(A + Ma^2 - C)\omega^2 - Mga]\varphi = 0. \tag{7}$$

For the value of the angular velocity equal to the bifurcation angular velocity, these equations must admit a solution different from the trivial one

Fig. 4

$$\alpha = 0, \quad \varphi = 0.$$

In this case the determinant of system (7) must be equal to zero. As a result we arrive at the equation, with respect to the unknown ω^2,

$$\omega^4 - \frac{g}{l}\left[l + \frac{Ma(l+a)}{A-C}\right]\omega^2 + \frac{Mga}{A-C}\frac{g}{l} = 0. \tag{8}$$

For an elongated body $A > C$. In this case the two essentially distinct roots of this equation are precisely the bifurcation values of the angular velocity. Comparison of the calculated numerical values of the angular velocity of bifurcation with the experimental ones showed their agreement.

The original equations (6) make it possible to find the angles α and φ as functions of the angular velocity ω. An approximate graph of these functions is shown in Figs. 4a and 4b. The experimentally measured values of the angles α and φ also proved to be in agreement with these graphs.

If the body is such that $A < C$, then the form of stationary rotation corresponding to Fig. 3b becomes impossible. Correspondingly, equation (8) has in this case one real positive root.

A rigorous proof of the stability of the basic form of stationary rotation of a rigid body suspended on a string, by the Lyapunov–Chetayev method, was given by E. P. Morozova ([2]). The study, by the same methods, of the stability of the forms of stationary motion that appear in the branching of the basic form is the subject of the paper by M. E. Temchenko ([3]).

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Note: Figure translations are in progress. See original paper for figures.