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Abstract

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FIELD THEORY OF LOCAL HYPERCONES IN X_n

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1. The field theory of local conical surfaces in X_n is of interest in connection with applications of differential geometry to the calculus of variations and to the theory of first-order partial differential equations with one unknown function. The field theory of local conical surfaces was first constructed in three-dimensional metric Euclidean space by C. Lee and G. Scheffers ([3]). Later this theory was considered by V. V. Wagner ([4]). From the point of view of applications, the assumption that the space is metric Euclidean is not expedient, since the study of a variational problem and of a differential equation specified up to arbitrary transformations of variables by means of a group of motions is of no special interest.

In 1948 V. V. Wagner ([1]) constructed the field theory of local conical surfaces in X_3 and considered its applications to the theory of differential equations and to the calculus of variations. In the present work the field theory of local hypercones in X_n for $n > 4$ is considered.

2. A central hypercone in a central E_n is uniquely determined by specifying an $(n-2)$-dimensional surface that is its director. It is obvious that the director is not determined uniquely. A hypercone is called pseudoregular if it admits, as its director, a regular $(n-2)$-dimensional surface. We shall consider only pseudoregular hypercones.

It is expedient to specify the director by parametric equations

$$x^\alpha = l^\alpha(\eta^a) \quad (a, b, \dots, e = 1, \dots, n-2). \tag{1}$$

Assuming that the parameters η^a may be subjected to transformations

$$\eta {a'}^a = \varphi^a(\eta^e), \tag{2}$$

where φ^a are regular functions of class v, we may regard the director as an $(n-2)$-dimensional Weyl–Whitehead space X_{n-2}. In this X_{n-2} the W-tensor densities $\mathfrak{G}_{ba}, \mathfrak{A}_{cba}$ of weight $-\dfrac{2}{n-2}$, the tensor h_{ba}, and the object of affine connection G^c_{ba} are defined invariantly.

As V. V. Wagner ([5]) showed, an $(n-2)$-dimensional regular surface in a central E_n is determined, up to an arbitrary centro-affine transformation, by specifying in X_{n-2}, for $n > 4$, the W-tensor densities $\mathfrak{G}_{ba}, \mathfrak{A}_{cba}$ and the contracted object of affine connection G_a, while for $n = 4$ the tensor h_{ba} is also adjoined to them.

A director of a hypercone for which the conditions

$$\nabla_a \mathfrak{M} = 0, \tag{3}$$

are satisfied,

where $\mathfrak{M}$ is a scalar W-density of weight $\dfrac{2}{n-2}$, which is a function of the W-tensor densities $\mathfrak{G}_{ba}$ and $\mathfrak{A}_{cba}$, we shall call a **normalized with the aid of the W-density $\mathfrak{M}$**. Conditions (3) define a one-parameter family of normalized directrices obtained from one another by means of a similarity transformation with center of similitude at the center E_n. It is not hard to see that, for a normalized directrix, the relations

$$G_a = \partial_a \ln \mathfrak{M}, \tag{4}$$

hold, whence follows the theorem:

Theorem 1. *A central pseudoregular hypercone in a central E_n is determined, up to an arbitrary central-affine transformation, by prescribing in X_{n-2}, for $n > 4$, the W-tensor densities $\mathfrak{G}_{ba}$ and $\mathfrak{A}_{cba}$, and for $n = 4$ the tensor h_{ba} is added to them.*

3. Defining local hypercones of a field by prescribing their normalized directrices, we reduce the field theory of local hypercones in X_n to the field theory of local $(n-2)$-dimensional normalized surfaces

$$x^\alpha = l^\alpha(\xi^\beta, \eta^e), \tag{5}$$

given up to arbitrary local similarity transformations

$$'x^\alpha = \frac{x^\alpha}{\sigma(\xi^\beta)}. \tag{6}$$

Assuming that the admissible transformations of the parameters η^a are determined by the equations

$$'\eta^\alpha = \varphi^a(\xi^\alpha, \eta^e) \left(\det \left\| \frac{\partial \varphi^a}{\partial \eta^b} \right\| \neq 0\right), \tag{7}$$

where φ^a are arbitrary functions, continuously differentiable a sufficient number of times with respect to the variables ξ^α and η^a, we may regard this field as a composite manifold $X_{n-2}(X_n)$ with the linear affine connection (2). In this $X_{n-2}(X_n)$, n fields of W-densities $\mathfrak{n}_a, m, \mathfrak{n}$ of weights respectively $-\dfrac{2}{n-2}$, 0, $-\dfrac{2}{n-2}$ are invariantly defined:

$$\mathfrak{n}_a = \mathfrak{n}_{a\alpha}(\xi^\beta, \eta^e) d\xi^\alpha; \qquad m = m_\alpha(\xi^\beta, \eta^e) d\xi^\alpha; \qquad \mathfrak{n} = \mathfrak{n}_\alpha(\xi^\beta, \eta^e) d\xi^\alpha. \tag{8}$$

These W-densities satisfy the following system of differential equations in total differentials:

$$\overset{2}{\nabla}_b \mathfrak{n}_a = -\mathfrak{G}_{ba} m + v_{ba} \mathfrak{n}; \qquad \overset{2}{\nabla}_a m = \mathfrak{H}^{b}_{.a} \mathfrak{n}_b + \mathfrak{W}_a \mathfrak{n}; \qquad \overset{2}{\nabla}_a \mathfrak{n} = \mathfrak{n}_a, \tag{9}$$

where $\mathfrak{H}^{b}_{.a} = \mathfrak{G}^{bc} h_{ca}$; $\overset{2}{\nabla}_c$ is the symbol of covariant differentiation with respect to the connection with coefficients

$$\overset{2}{G}{}^c_{ba} = G^c_{ba} + \mathfrak{G}^{cd} \mathfrak{A}_{bad}, \tag{10}$$

and v_{ba} and $\mathfrak{W}_a$ are known functions of the W-tensor densities $\mathfrak{G}_{ba}$ and $\mathfrak{A}_{cba}$. Under local similarity transformations (6), the W-densities $\mathfrak{n}_a$, m, and $\mathfrak{n}$ are multiplied by $\sigma(\xi^\alpha)$.

Suppose that in $X_{n-2}(X_n)$, by means of the Pfaff equations

$$\delta\eta^a = d\eta^a + \Gamma^a = 0 \qquad (\Gamma^a = \Gamma^a_\alpha(\xi^\beta, \eta^e) d\xi^\alpha) \tag{11}$$

a linear connection, invariant with respect to the local similarity transformations (6), is defined, and let

$$\Gamma^\alpha = \nu^{ba} \eta_b + \nu^a_{n-1} m + \nu^a_n \eta; \tag{12}$$

$$[\partial\eta_e] = \mathfrak{R}_e^{\cdot ba} [\eta_b \eta_a] + 2\mathfrak{R}^{\cdot a}_{e\,(n-1)} [\eta_a m] + 2\mathfrak{R}^{\cdot a}_{e\,(n)} [\eta_a \eta] + 2\mathfrak{R}_{e\,(n-1,n)} [m\eta]; \tag{13}$$

$$[\partial\eta] = \Lambda^{ba} [\eta_b \eta_a] + 2\Lambda^a_{(n-1)} [\eta_a m] + 2\Lambda^a_{(n)} [\eta_a \eta] + 2\Lambda_{(n-1,n)} [m\eta], \tag{14}$$

where the square brackets denote Cartan exterior multiplication. Introducing the notation

$$[D\eta_e] = \mathfrak{V}_e^{\cdot ba}[\eta_b\eta_a] + 2W_{e\,(n-1)}^{\cdot a}[\eta_a m] + 2\mathfrak{V}_{e\,(n)}^{\cdot a}[\eta_a\eta] + 2W_{e\,(n-1,n)}[m\eta]; \tag{15}$$

$$[Dm] = J^{ba}[\eta_b\eta_a] + 2J_{n-1}^a[\eta_a m] + 2J_n^a[\eta_a\eta] + 2J_{(n-1,n)}[m\eta]; \tag{16}$$

$$[D\eta] = \mathfrak{V}^{ba}[\eta_b\eta_a] + 2V_{n-1}^a[\eta_a m] + 2\mathfrak{V}_n^a[\eta_a\eta] + 2V_{(n-1,n)}[m\eta], \tag{17}$$

where D is the symbol of the absolute basis differential, we define the invariant connection by the conditions:

$$\mathfrak{V}^{ba} = 0; \qquad V_{n-1}^a = 0; \qquad W_c = 0; \qquad D_{n-1}\mathfrak{G}^{ba} = 0; \qquad D_{n-1}\mathfrak{M} = 0, \tag{18}$$

where the differential operator D_{n-1} is defined by the symbolic equality

$$D = \eta_a\vartheta^a + mD_{n-1} + \eta\vartheta_n. \tag{19}$$

After some computations we obtain

$$\nu^{ba} = \Lambda^{ba} + \frac{1}{2}\,\iota^\omega(\partial_\omega\mathfrak{G}^{ba} - \mathfrak{G}^{ba}\partial_\omega \ln\mathfrak{M}) + \Lambda_{(n-1)}^e\partial_c\mathfrak{G}^{ba} - \mathfrak{G}^{ba}\partial_c\ln\mathfrak{M}; \tag{20}$$

$$\nu_n^a = -2\Lambda_{(n-1)}^a; \tag{21}$$

$$\nu_n^a = 2\mathfrak{G}_{(n-1,n)}^{aa}\left(\mathfrak{R}_c + \Lambda_{(n-1)}^e\vartheta_{ec}\right). \tag{22}$$

The invariant connection can be defined not in a unique way, at least because the W-density $\mathfrak{M}$ is chosen with great arbitrariness.

4. If the field of local hypercones in X_n is constant in some domain T of the basis X_n, then there exists such a system of coordinates in X_n, a geometric domain containing T, and such a field of local coordinate systems in this domain that the equations of the normalized directions of the local hypercones of the field in the domain T are written in the form

$$x^\alpha = \delta(\xi^\beta)l^\alpha(\eta^\varepsilon). \tag{23}$$

Theorem 2. *The field of local $(n-2)$-dimensional normalized surfaces in X_n, for $n \geqslant 4$, can be transformed in the domain T by means of local similarity transformations into a constant field if and only if in the domain T the conditions*

$$[\partial\Gamma^a] - [\Gamma^e \partial_e \Gamma^a] = 0; \qquad D\mathfrak{G}_{ba} = 0; \qquad D\mathfrak{R}_{cba} = 0; \tag{24}$$

are satisfied, and for $n = 4$ also the condition

$$Dh_{ba} = 0. \tag{25}$$

The necessity of these conditions may be regarded as obvious. To prove sufficiency, choose local coordinate systems in the domain T in such a way that all the Pfaffians of the connection Γ^a vanish. It is not difficult to see that from conditions (24) and (25) it follows that the W-tensor densities $\mathfrak{S}_{ba}$, $\mathfrak{A}_{cba}$, and the tensor h_{ba} in the domain T do not depend on the coordinates ξ^α of the point of the base X_n. Therefore the solution of system (9) can be written in the form

$$\underset{\alpha}{\mathfrak{n}}_a = \ast\underset{\alpha}{\mathfrak{n}}_a(\eta^e)\,\overset{\alpha}{\omega}(\xi^\beta); \qquad \underset{\alpha}{m} = \ast\underset{\alpha}{m}(\eta^e)\,\overset{\alpha}{\omega}(\xi^\beta); \qquad \underset{\alpha}{\mathfrak{n}} = \ast\underset{\alpha}{\mathfrak{n}}(\eta^e)\,\overset{\alpha}{\omega}(\xi^\beta), \tag{26}$$

where $(\ast\underset{\alpha}{\mathfrak{n}}_a, \ast\underset{\alpha}{m}, \ast\underset{\alpha}{\mathfrak{n}})$ are n linearly independent solutions of system (9), and

$$\overset{\alpha}{\omega} = e^\alpha_\beta(\xi^\gamma)\,d\xi^\beta.$$

It may be shown that

$$\overset{\alpha}{\omega} = \theta_{(\alpha)}\,d\varphi^{(\alpha)}; \qquad \theta_\alpha = \sigma(\xi^\beta)\psi_\alpha(\varphi^\alpha), \tag{27}$$

where φ^α is a function of the variables ξ^β, and summation over the index α is not implied. Introducing in X_n the new variables

$$\tilde{\xi}^\alpha = \int \psi_{(\alpha)}(\varphi^\alpha)\,d\varphi^{(\alpha)}, \tag{28}$$

we obtain

$$\mathfrak{n}_a = \sigma\, \ast\underset{\alpha}{\mathfrak{n}}_a\, d\tilde{\xi}^\alpha; \qquad m = \sigma\, \ast\underset{\alpha}{m}\, d\tilde{\xi}^\alpha; \qquad \mathfrak{n} = \sigma\, \ast\underset{\alpha}{\mathfrak{n}}\, d\tilde{\xi}^\alpha, \tag{29}$$

whence it follows that, in the new coordinate system, the equations of the $(n-2)$-dimensional normalized surfaces of the field will have the form

$$l^\alpha(\tilde{\xi}^\beta, \eta^e) = \frac{1}{\sigma(\xi^\beta)}\, l^\alpha(\eta^e). \tag{30}$$

Applying the results obtained to the field of local hypercones in X_n, we shall have:

Theorem 3. *The field of local hypercones in X_n for $n \geqslant 4$ will be constant in the domain T of the base X_n if and only if in the domain T conditions (24) are satisfied, and for $n = 4$ also condition (25).*

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Note: Figure translations are in progress. See original paper for figures.