# MATHEMATICS

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## Abstract

## Full Text

MATHEMATICS

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# ON ELLIPTIC SYSTEMS OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Let there be given an elliptic system of linear partial differential equations

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + A_1u_x + B_1u_y + C_1u = 0, \tag{1}$$

where $A, B, C, A_1, B_1, C_1$ are given continuous real square matrices of order $n$ in some domain $D_1$ of the variables $x, y$, and $u = (u_1, u_2, \dots, u_n)$ is the vector sought.

If the coefficients of the principal part of system (1) in the domain $D_1$ satisfy the condition of positive definiteness of the quadratic form—the Somigliana condition ($^1$):

$$\eta A\eta + \eta B\xi + \xi B\eta + \xi C\xi \geq 0\,*, \tag{2}$$

where $\eta = (\eta_1, \eta_2, \dots, \eta_n)$, $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, then for the Dirichlet problem—to find, in a finite simply connected domain $D \subset D_1$, a solution $u(x, y)$ of system (1), taking prescribed continuous values $f$ on the boundary $\Gamma$—in the case of sufficient smoothness of $\Gamma$ and of the coefficients of this system, the following Fredholm alternative holds.

**Fredholm alternative.** *The nonhomogeneous problem is always solvable if the corresponding homogeneous problem has only the trivial solution* ($^{2,4}$)$^{**}$.

This alternative indicates the importance of conditions sufficient for the homogeneous Dirichlet problem to have only the trivial solution.

For a system of the form

$$Lu = (Au_x + Bu_y)_x + (Bu_x + Cu_y)_y + A_1u_x + B_1u_y + C_1u = 0 \tag{3}$$

in the case when $A_1$ and $B_1$ are symmetric and (2) is satisfied, one such condition is easily established.

Indeed, applying Green-Ostrogradsky' s formula (it is assumed that the contour $\Gamma$, the coefficients of system (3), and the vector sought $u(x, y)$ satisfy the conditions ensuring the applicability of this formula), and taking into account that $u$ is a solution of the homogeneous problem, from the identity

$$(uAu_x + uBu_y + \tfrac{1}{2}uA_1u)_x + (uBu_x + uCu_y + \tfrac{1}{2}uB_1u)_y =$$

$$= uLu + u_{xAu}x + u_{xBu}y + u_{yBu}x + u_{yCu}y + \tfrac{1}{2}u(A_{1x} + B_{1y} - 2C_1)u$$

we shall have

$$\iint_D \left[u_{xAu}x + u_{xBu}y + u_{yBu}x + u_{yCu}y + \right.$$
$$\left. + \frac{1}{2}u\left(A_{1x} + B_{1y} - 2C_1\right)u\right] dx\,dy = 0. \qquad (4)$$

Assuming now that the condition

$$\eta(A_{1x} + B_{1y} - 2C_1)\eta \geqslant 0, \qquad (x, y) \in D \qquad (5)$$

is satisfied,* we obtain, by virtue of (2) and (4), $u(x, y) = 0$.

In the case when $A_1 = B_1 = 0$, condition Pini (5) follows from (5). We note that condition (5) is directly generalized to systems of the form (3) in the case of many independent variables.

In [5] it is proved that, if the matrix $-C_1$ is positive definite, then the length $R = (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}$ of a vector $u$, regular in the domain $D$, which is a

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* The scalar product of two vectors $u(u_1, u_2, \ldots, u_n),\ v = (v_1, v_2, \ldots, v_n)$

is understood to be the sum

$$\sum_{i=1}^{n} u_i v_i.$$

** In (2) this alternative is established under the assumption that $A = C = E$

is the identity diagonal matrix, $B = 0$, and the remaining coefficients are analytic functions of their arguments (see also (3)).

solution of the system (3) for $A = C = E,\ B = A_1 = B_1 = 0$, cannot have a maximum in the interior of the domain $D$. This assertion is easily found, under positive definiteness of $-C_1$, also in the case when $A = \alpha(x,y)E,\ B = 0,\ C = \beta(x,y)E,\ A_1 = \alpha_1(x,y)E,\ B_1 = \beta_1(x,y)E$, where $\alpha, \beta, \alpha_1, \beta_1$ are given functions (scalar quantities), with $\alpha > 0,\ \beta > 0$.

Indeed, suppose that $R(x,y)$ has a maximum at an interior point $(x,y)$ of the domain $D$. On the one hand, at this point we must have

$$R_x = R_y = 0, \tag{6}$$

$$\alpha R_{xx} + \beta R_{yy} \leqslant 0. \tag{7}$$

But, on the other hand, by virtue of (3), (6), and the positive definiteness of the matrix $-C_1$, the inequality

$$\alpha R_{xx} + \beta R_{yy} = (-uC_1 u + \alpha u_x^2 + \beta u_y^2)/R > 0$$

holds, which contradicts condition (7). This fact, obviously, also holds in the case of many independent variables for the corresponding equation.

A simple example of an elliptic system

$$u_{1xx} - u_{1yy} - 2u_{2yy} = 0, \qquad 2u_{1xx} + u_{2xx} - u_{2yy} = 0$$

shows that, although condition (2) (or, as it is also called, the condition of strong ellipticity) is not satisfied, nevertheless the Dirichlet problem always has a solution, and moreover a unique one.

On the other hand, for the elliptic system considered in [6]

$$u_{1xx} - u_{1yy} - 2u_{2xy} = 0, \qquad 2u_{1xy} + u_{2xx} - u_{2yy} = 0,$$

which is also not strongly elliptic, the Dirichlet problem is, in general, impossible. Indeed, let $D$ be the circular domain $|z| < 1$ of the plane

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* Condition (5) in the case when $A = C = E,\ B = 0$, appears in our paper [7]. In the same paper it is shown that conditions (5) and

$$\eta(G_s - 2H + A_1 \cos(\widehat{N,x}) + B_1 \cos(\widehat{N,y}))\eta \geqslant 0, \qquad (x,y) \in \Gamma$$

are sufficient for the uniqueness of the solution of the more general problem with boundary condition

$$\frac{du}{dN} + G\frac{du}{ds} + Hu = f,$$

where $N$ is the inward normal, and $G$ and $H$ are prescribed continuous matrices, with $G$ diagonal.

of the complex variable $z = x + iy$, on whose boundary $\Gamma$ the values $u_1 = f_1,\ u_2 = f_2$ are prescribed. Since all regular solutions of this system are represented in the form $u_1 + iu_2 = z\varphi(z) + \psi(\bar{z})$, where $\varphi$ and $\psi$ are arbitrary holomorphic functions of the variable $z$, and $\bar{z} = x - iy$, it is clear from this that, for the solvability of the Dirichlet problem, it is necessary that $tf = (f_1 + if_2)t$ be the boundary value of a function holomorphic inside the domain $D$. If this condition is satisfied, then the solution of the problem will have the form

$$u_1 + iu_2 = (1 - z\bar{z})\psi(\bar{z}) + \frac{\bar{z}}{2\pi i}\int_\Gamma \frac{tf(t)\,dt}{t - z},$$

where $\psi$ is an arbitrary holomorphic function.

At first glance it may seem strange that, for the system just considered, the boundary problem $u_1 = f_1,\ u_{1x} - u_{2y} = f_2$ is always possible, and moreover the homogeneous problem admits the solution $u_1 = 0,\ u_2 = ax + b$.

Let us give another curious example of a second-order system

$$x\Delta u_1 + y\Delta u_2 - 2(u_{1x} + u_{2y}) = 0, \qquad y\Delta u_1 - x\Delta u_2 + 2(u_{2x} - u_{1y}) = 0,$$

elliptic everywhere except at the point $z = 0$ (at this point a degeneracy of order takes place), and possessing the following property: in any circular domain for which $z = 0$ is not an interior point, the Dirichlet problem always has a (and, moreover, unique) solution, while if $z = 0$ is an interior point of the circle, then although uniqueness does not hold, this problem is always solvable. For example, in the disk $|z| < 1$, the solution of the Dirichlet problem $u_1 + iu_2 = f$ (on $\Gamma$) is given by the formula

$$u_1 + iu_2 = \alpha(1 - z\bar{z}) + \frac{z\bar{z}}{2\pi i}\int_\Gamma \frac{f\,dt}{t - z} - \frac{1}{2\pi i}\int_\Gamma \frac{f\,dt}{t} - \frac{1}{2\pi i}\int_\Gamma \frac{\bar{f}\,dt}{t - z},$$

where $\alpha$ is an arbitrary constant.

Suppose now that there is an elliptic system

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0 \tag{8}$$

with constant coefficients. Let $\alpha_1, \alpha_2, \dots, \alpha_\mu$; $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_\mu$ denote the roots of the characteristic equation $\det|A + 2B\lambda + C\lambda^2| = 0$, and $k_1, k_2, \dots, k_\mu$ their multiplicities, respectively.

All regular solutions of system (8) can be represented in the form

$$u = \operatorname{Re} \sum_{j=1}^{p} \sum_{l=1}^{k_j} \sum_{m=0}^{l-1} C_{lm}^{(j)} \, \bar{z}_j^{\,m} \, \varphi_{jl}^{(m)}(z_j), \tag{9}$$

where the $\varphi_{jl}$ are arbitrary holomorphic functions of the complex variables $z_j = x + \alpha_j y$, respectively; the upper index $m$ of the functions $\varphi_{jl}$ indicates the order of differentiation with respect to $z_j$, while the $C_{lm}^{(j)}$ are completely determined constant vectors, which are expressed solely through the coefficients of system (8), and, moreover, for finding $C_{lm}^{(j)}$ it is necessary to solve systems of linear algebraic equations.

Formula (9) makes it possible to reduce the investigation of any linear boundary-value problem (Dirichlet, Poincaré, etc.) to an equivalent boundary-value problem in the theory of holomorphic functions.

It is clear from this that, for the boundary-value problems mentioned, one could not, in general, expect alternatives of Fredholm type or of Noether type to hold.

In the case where the characteristic equation has $n$-fold roots $\alpha_1, \bar{\alpha}_1$, the investigation of these problems is considerably simplified.

We note that, using formula (9), one can explicitly construct singular solutions of system (8). For this it suffices to replace the functions $\varphi_{lm}$ by analytic functions with the required singularity.

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*Note: Figure translations are in progress. See original paper for figures.*

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