INDUCTIVE AND PROJECTIVE LIMITS WITH COMPLETELY CONTINUOUS MAPPINGS

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INDUCTIVE AND PROJECTIVE LIMITS WITH COMPLETELY CONTINUOUS MAPPINGS

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In the work of Sebastião e Silva (^1) two classes of locally convex spaces are considered, encompassing the majority of spaces of analytic, infinitely differentiable, and generalized functions now used in analysis. In the present note further properties of these classes are reported and, in addition, some properties are given of the projective limit of an arbitrary inverse spectrum of locally convex spaces with respect to completely continuous mappings.

Definition 1. Let $E_1, E_2, \ldots$ be an expanding sequence of locally convex spaces (i.e. each of them is a vector subspace of the next), and let the embedding (i.e. the identity mapping) of each into the next be continuous. The **inductive limit** of this sequence of spaces (with respect to the indicated embeddings) is their union E, endowed with the strongest locally convex topology in which all embeddings $E_n \to E$ are continuous.

Definition 2. Let E and G be locally convex spaces. A linear mapping of E into G is called **completely continuous** if the image of some neighborhood of zero in E has a bicompact closure in G.

Definition 3. A **regular** sequence of locally convex spaces is any expanding sequence of them with completely continuous embeddings. A **space** (LN^*), according to Sebastião e Silva, is the inductive limit of a regular sequence of normed spaces.

Theorem 1. *The inductive limit of any regular sequence of locally convex spaces is a space* (LN^*).

Theorem 2. *The quotient space of a space* (LN^*) *by its closed subspace is a space* (LN^*).

Theorem 3. *A closed subspace of a space* (LN^*) *is a space* (LN^*).

Definition 4. If $E_1, E_2, \ldots$ is a sequence of locally convex spaces and π_m^n, for all m and $n > m$, are continuous linear mappings of E_n into E_m such that $\pi_m^p = \pi_m^n \pi_n^p$ for $m < n < p$, then the spaces E_n are said to form an **inverse spectrum** with respect to the mappings π_m^n. Let E be the totality of all possible "threads" $x = (x_n)$, i.e. sequences $x_1, x_2, \ldots$, where each $x_n \in E_n$

and $x_m = \pi_m^n(x_n)$ for all m and $n > m$. E is a vector space with respect to "coordinatewise" addition and multiplication by scalars, and $x \to x_n = \pi_n(x)$ is a linear mapping ("projection") of E into E_n. The **projective limit** of the sequence $E_1, E_2, ...$ with respect to the mappings π_m^n is the space E, endowed with the weakest topology for which all projections π_n are continuous.

Definition 5. A space (M^*) is called, after Sebastião e Silva, the projective limit of a sequence of normed spaces forming an inverse spectrum with respect to completely continuous mappings.

Theorem $1'$**.** *The projective limit of any sequence of locally convex spaces forming an inverse spectrum with respect to completely continuous mappings is a space* (M^*)*.*

Theorem $2'$**.** *A closed subspace of a space* (M^*) *is a space* (M^*)*.*

Theorem $3'$**.** *The quotient space of a space* (M^*) *by its closed subspace is a space* (M^*)*.*

Thus, each of the classes (LN^*) and (M^*) is closed under passage to quotient spaces and subspaces.

Definition 6. We shall say that locally convex spaces E, G form a **strongly dual pair** if each of them is isomorphic to the space dual to the other, endowed with the strong topology.

As Sebastião e Silva showed, the spaces (M^*) and (LN^*) are reflexive, and if E is a space (M^*) (respectively (LN^*)), then E', endowed with the strong topology, is a space (LN^*) (respectively (M^*)).

Theorem 4. *Let* E, G *be a strongly dual pair, and suppose that one of them is a space* (M^*) *and, hence, the other is a space* (LN^*)*. Let, further,* F *be an arbitrary closed subspace in* E*, and let* H *be its orthogonal complement, i.e. the annihilator in* G*. Then* $F, G/H$ *and* $E/F, H$ *are also strongly dual pairs.*

Thus, the strong duality established by Sebastião e Silva for the spaces (M^*) and (LN^*) extends also to their quotient spaces and subspaces.

At the basis of the proof of Theorems 2, 3, $2'$, $3'$, and 4 lies the following proposition:

A space (LN^*) *contains "regular" sequences of bicompact subsets and is the free union of every such sequence.*

Here a sequence of bicompact subsets $K_1, K_2, ...$ of a locally convex space E is called **regular** if the following conditions are satisfied: 1) all K_n are absolutely convex; 2) each K_n is contained algebraically inside K_{n+1}; 3) $E = \bigcup K_n$; and 4) each K_n is bicompact in the linear span F_{n+1} of the set K_{n+1}, considered as a normed space with unit ball K_{n+1}. The assertion that E is the **free union** of the sets K_n means that a set $M \subset E$ is closed if and only if each $M \cap K_n$ is closed in K_n.

Definition 7. By a **space of type** (c) we mean a locally convex space admitting no stronger locally convex topology under which all absolutely convex sets that are bicompact in the original topology would remain bicompact.

Theorem 5. *A continuous linear mapping of a space* (LN^*) *onto a space of type* (c) *is open.*

This theorem is stronger than the theorem obtained from Grothendieck's general "Theorem B" in ([2]), when the range space (LF) is a space (LN^*). Indeed, every Grothendieck space of type (β) is a space of type (c), whereas there exist spaces of type (c) which are not spaces of type (β).

As is known, Definition 4 extends directly to any inverse spectrum $(E_\alpha, \pi_\alpha^\beta)$, where α, β range over a directed set of indices A, the E_α are locally convex spaces assigned to all $\alpha \in A$, and π_α^β $(\beta > \alpha)$ are continuous linear mappings of E_β into E_α,

satisfying the condition $\pi_\alpha^\gamma = \pi_\alpha^\beta \pi_\beta^\gamma$ for $\alpha < \beta < \gamma$. Dual to it is the definition of the inductive limit of a direct spectrum.

Definition 8. Let A be a directed set of indices; with each $\alpha \in A$ associate a locally convex space E^α, and with each pair $\alpha, \beta \in A$, where $\beta > \alpha$, a continuous linear mapping π_β^α of the space E^α into E^β, moreover

$$\pi_\gamma^\alpha = \pi_\gamma^\beta \pi_\beta^\alpha$$

for $\alpha < \beta < \gamma$. Then one says that the E^α form a direct spectrum with respect to the mappings π_β^α, or, more briefly, that a direct spectrum $(E^\alpha, \pi_\beta^\alpha)$ is given. Let E be the set of all possible "fibers," i.e. sets $(\pi_\beta^\alpha x^\alpha)$, where α may be any fixed index from A, x^α any element of E^α, and β runs through all indices $> \alpha$, with fibers that merge from some point on being identified. E is a vector space with respect to the "coordinatewise" addition and multiplication by scalars, and $x^\alpha \to x = (\pi_\beta^\alpha x^\alpha) = \pi^\alpha(x^\alpha)$ is a linear mapping ("injection") of E^α into E. The **inductive limit** of the direct spectrum $(E^\alpha, \pi_\beta^\alpha)$ is the space E endowed with the strongest locally convex topology for which all injections π^α are continuous.

In the particular case when A is the sequence of natural numbers with its natural order, E^m is a vector subspace of E^n for all m and $n > m$, and π_n^m is the embedding of E^m into E^n, E is naturally identified with $\bigcup E^n$, while the injection π^n is the embedding of E^n into E, and we arrive at Definition 1.

Theorem 6. Let E be the projective limit of the inverse spectrum $(E_\alpha, \pi_\alpha^\beta)$, and suppose that for each α there exists $\beta > \alpha$ such that π_α^β is completely continuous.

Then: 1) E is the projective limit of an inverse spectrum of Banach spaces and therefore is a complete locally convex space; 2) every closed bounded set in E is bicompact; 3) E', endowed with the strong topology, is isomorphic to the inductive limit of the direct spectrum $(E^\alpha, \pi_\beta^\alpha)$, where E^α is the strong conjugate of E_α, and π^α_β is the mapping of E^α into E^β conjugate to π^β_α (under the assumption that $\pi_\alpha E$ is dense in E_α for every α).

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References

[1] J. Sebastião e Silva, Rend. mat. pura e appl., **14**, 388 (1955) (Russian translation: *Matematika*, **1**, 60 (1957)). [2] A. Grothendieck, Mem. Am. Math. Soc., **16** (1955).

Note: Figure translations are in progress. See original paper for figures.