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Abstract

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SOME TAUBERIAN THEOREMS FOR GENERALIZED FOURIER TRANSFORMS

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Let V_n $(n \geq 0)$ denote the class of complex-valued functions $\alpha(\lambda)$ of bounded variation on every finite interval, satisfying the condition

$$\alpha_n^*(\lambda) \equiv \lim_{|a|\to\infty} |a|^{-n} \operatorname*{Var}_{a}^{a+\lambda}\{\alpha(\mu)\} < \infty. \tag{1}$$

Every function $\alpha(\lambda) \in V_n$ has a Bochner–Stieltjes transform of order $^*[n+2]$

$$E_{[n+2]}(x;\alpha) = \int_{\infty}^{-\infty} (i\lambda)^{-[n+2]} \left\{ e^{i\lambda x} - \chi_{-1,1}(\lambda) \sum_{\nu=0}^{[n+1]} \frac{(i\lambda x)^\nu}{\nu!} \right\} d\alpha(\lambda),$$

where $\chi_{a,b}(\lambda)$ is the characteristic function of the interval $[a,b]$.

The differential properties of the function $E_{[n+2]}(x;\alpha)$ are evidently determined by the behavior of the function $\alpha(\lambda)$ for large $|\lambda|$. For example, if as $|\lambda| \to \infty$ one has $\alpha(\lambda) = \lambda^{n+1} + O(|\lambda|^{n-1-\varepsilon})$ $(\varepsilon > 0)$, then the function $E_{[n+2]}(x;\alpha)$ possesses an absolutely continuous derivative of any order ** less than $[n+1] - n$.

More difficult is the question of what additional conclusions (in comparison with (1)) about the behavior of the function $\alpha(\lambda)$ as $|\lambda| \to \infty$ can be drawn if it is known that the function $E_{[n+2]}(x;\alpha)$ possesses certain differential properties on the whole axis or on some finite interval. This question, in connection with the investigation of the asymptotic properties of spectral functions of differential operators, was considered in a number of works by B. M. Levitan ([1–3]) and V. A. Marchenko ([4]). The present note is also devoted to it.

The study of the function $\alpha(\lambda)$ as $|\lambda| \to \infty$, in essence, consists in the study of the integral

$$\int_{-\infty}^{\infty} d\alpha(\lambda). \tag{2}$$

Since the integral (2), generally speaking, is divergent, one has to apply to it one or another summation method, i.e., instead of (2) investigate, as $N \to 0$, the asymptotics of the convergent integral

$$\int_{-\infty}^{\infty} T(N,\lambda)\, d\alpha(\lambda) \qquad (N > 0), \tag{3}$$

* $[x]$ is the integer part of the number x.

** Concerning the definition of derivatives of fractional orders, see, for example, ([5]), pp. 221–223.

formed by means of some kernel $T(N,\lambda)$. The kernel may be taken, for example, to be $(1 - N^2\lambda^2)^k \chi_{-M,M}(\lambda)$ ($k \geqslant 0;\ M = N^{-1}$).

If in some interval $(-h, h)$ the function $E_{[n+2]}(x;\alpha)$ is a polynomial of degree not higher than $[n+1]$, then, as B. M. Levitan showed ([3]), as $N \to 0$,

$$\int_{-M}^{M} (1 - N^2\lambda^2)^k\, d\alpha(\lambda) = O(N^{k-n}). \tag{4}$$

Consequently, for $k > n$ there exists a limit* of the integral (4), which, generally speaking, is not true for $k < n$. In this sense the value $k = n$ is critical. The critical value singles out, from the scale of one-type summability methods, the method most precisely corresponding to the given rate of divergence of the integral (2).

The question arises whether, for kernels $T(N,\lambda)$ of general form, one cannot introduce a parameter with respect to which it would make sense to speak of a critical value.

If the number n is an integer, then, as follows from the work of V. A. Marchenko ([4]), such a parameter is the maximal number k of derivatives of the kernel with respect to λ, the last of which has bounded variation, and the critical value will** still be $k = n$. Therefore, in passing to nonintegral n, it is natural to introduce into consideration derivatives of fractional order of the kernel. At the same time one has to take into account derivatives of fractional order of the function $E_{[n+2]}(x;\alpha)$.

The results obtained by the author are a generalization of Theorems 4.1, 4.2 ([4]) to the case of nonintegral n, k. The difficulties arising here are due to the nonlocal character of the operation of differentiation of nonintegral order and to its asymmetry with respect to the change of sign of the argument.

For simplicity of formulation we shall restrict ourselves to kernels of the form

$$T(N,\lambda) = T(N\lambda), \tag{5}$$

where $T(\lambda)$ is a finite function. For definiteness let $T(\lambda) = 0$ for $|\lambda| \geqslant 1$.

We shall say that a function $f(x)$, summable on a finite interval $[a, b]$, belongs to the class $D_s(a, b)$ $(s \geqslant 0)$, if for $s \geqslant 1$ the function $f(x)$ has on $[a, b]$ an absolutely continuous derivative of order $[s-1]$ and if the function

$$I(x; f) = \frac{1}{\Gamma([s+1]-s)} \int_{-\infty}^{x} (x-t)^{[s]-s} \chi_{a,b}(t) f^{([s])}(t)\, dt$$

is absolutely continuous. In this case the summable function $f^{(s)}(x) = I'(x; f)$ will be called the derivative of order s of $f(x)$.

Assume that:

A. $T(\lambda)$ is a function of bounded variation, belonging, together with the function $T(-\lambda)$, to some class $D_k(-\lambda_0, 0)$ $(\lambda_0 > 1)$, and is continuous for $k > 0$.

B. $T^{(k)}(\lambda)$, $\{T(-\lambda)\}^{(k)}$ are functions of bounded variation $(\lambda \leqslant 0)$.

C. The function $T(\lambda)$ is continuous and $[k]$ times differentiable at the point $\lambda = 0$.

Theorem 1. Let the function $E_{[n+2]}(x; \alpha)$ belong to some class $D_p(-h, h)$ $([n+2] - n \leqslant p \leqslant [n+2])$. Put

$$m = [n+2] - p, \qquad G(x) = E^{(p)}_{[n+2]}(x; \alpha) \chi_{-h,h}(x).$$

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* The equality of this limit to zero is immaterial here.
** At least for kernels of the form (5).

Then

$$\lim_{N \to 0} N^{n-k} \left| \int_{-\infty}^{\infty} T(N\lambda)\, d\alpha(\lambda) - \frac{1}{2\pi} \int_{-\infty}^{\infty} T(N\lambda)(i\lambda)^m E_G(\lambda)\, d\lambda \right| \leq$$

$$\leq C(1+k)^5 v_{n,k} h^{-k} \alpha^*_n \left(\frac{1}{h}\right), \tag{6}$$

where $E_G(\lambda)$ is the Fourier transform of the function $G(x)$; C is an absolute constant;

$$v_{n,k} = \int_{-\infty}^{0} |\lambda|^n \left(|dT^{(k)}(\lambda)| + \left| d_\lambda \{T(-\lambda)\}^{(k)} \right| \right).$$

The following propositions make it possible to give estimate (6) a simpler form.

Theorem 2. If $m + k + 1 > n$ and

$$\int_{-h}^{h} |x|^{-(m+k+1-n)} |G(x)|\, dx < \infty,$$

then, as $N \to 0$,

$$Z(N) = \frac{1}{2\pi} \int_{-\infty}^{\infty} T(N\lambda)(i\lambda)^m E_G(\lambda)\, d\lambda =$$

$$= \sum_{\nu=0}^{[k-n]} \frac{T^{(\nu)}(0)(-iN)^\nu}{\nu!\, \Gamma(-m-\nu)} \int_{-h}^{0} \frac{G(x)\, dx}{|x|^{m+\nu+1}} + o(N^{k-n}) \ * .$$

If, however, $m + k + 1 \leq n$, then

$$Z(N) = o(N^{k-n}).$$

Theorem 3. Suppose that in some interval $[-\varepsilon, \varepsilon]$ the function $G(x)$ is continuous and belongs to the class $D_k(-\varepsilon, \varepsilon)$. If the Fourier series of the function $G^{(k)}(x)$ converges, together with the conjugate trigonometric series, at the point $x = 0$, then, as $N \to 0$,

$$Z(N) = \sum_{\nu=0}^{[k-n]} \frac{T^{(\nu)}(0)(-iN)^\nu}{\nu!} G^{(m+\nu)}(0) + o(N^{k-n}),$$

where $G^{(m+\nu)}(0)$ is ** the sum of the Fourier series of the function

$$-\frac{d^{[m+\nu+1]}}{dx^{[m+\nu+1]}} \frac{1}{\Gamma([m+1]-m)} \int_{-h}^{x} (x-t)^{[m]-m} G(t)\, dt$$

at the point $x = 0$.

In the proofs of Theorems 1 and 2 an essential role is played by the asymptotic expansion of the confluent hypergeometric function.

Let us give an example showing that Theorems 1-3 make it possible to detect the critical value of the parameter k in the case of nonintegral n.

Denote by $g(x)$ $(-\pi < x < \pi)$ the summable function equal to zero on the intervals $(-\pi, \pi/4)$, $(\pi/2, \pi)$, and equal to $\sum_{\nu=2}^{\infty} e^{i\nu x} \ln^{-2} \nu$ on the interval $(\pi/4, \pi/2)$. Further denote by a_ν the complex Fourier coefficients of the function $g(x)$, and construct a step function $\alpha(\lambda)$ with jumps at the integer points $\lambda = \nu$, respectively equal to $(i\nu)^n a_\nu$, where n is a fixed number $(0 < n < 1)$.

* The sum on the right-hand side should be taken to be zero for $k < n$.

** The existence of the quantities $G^{(m+\nu)}(0)$ is established along the way.

Obviously, $\alpha(\lambda) \in V_n$, and $\alpha_n^*(\lambda) \equiv 0$. It is verified directly that $E_2(x;\alpha) \in D_{2-n}(-\pi,\pi)$ and that the function $E_2^{(2-n)}(x;\alpha)$ is infinitely differentiable in the interval $(-\pi/4,\pi/4)$. Therefore, from Theorems 1 and 3 there follows the existence of

$$\lim_{N\to 0}\int_{-M}^{M}(1-N|\lambda|)^n\,d\alpha(\lambda) \qquad (M=N^{-1}).$$

On the other hand, whatever k may be $(0\le k<n)$, the quantity

$$\int_{-M}^{M}(1-N|\lambda|)^k\,d\alpha(\lambda)=\sum_{|\nu|<M}(1-N|\nu|)^k(i\nu)^n a_\nu$$

has no limit as $N\to 0$. Otherwise, as $\nu\to\infty$, we would have had (([6]), pp. 146, 132)

$$i^n a_\nu + (-i)^n a_{-\nu} = o(\nu^{k-n}),$$

which is impossible, since

$$a_\nu = \begin{cases} 1/8\ln^2\nu, & (\nu = 2,3,4,\ldots),\\ 0, & (\nu = 1,0,-1,-2,\ldots).\end{cases}$$

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Note: Figure translations are in progress. See original paper for figures.