MATHEMATICS

1957

SovietRxiv

Abstract

Full Text

MATHEMATICS

E. A. TROITSKAYA

APPLICATION OF THE GENERAL THEORY OF APPROXIMATE METHODS TO THE STUDY OF THE PROBLEM OF DETERMINING EIGENVALUES AND EIGENVECTORS

(Presented by Academician V. I. Smirnov, 5 XI 1956)

We consider two completely continuous operators: A in a linear normed space X and $\bar{A}$ in a complete linear normed space $\bar{X}$, connected in the following way. In the space X there exists a subspace $\widetilde{X}$, isomorphic to $\bar{X}$. The isomorphism is carried out by means of a linear operation φ_0, which has an inverse φ_0^{-1} and admits an extension φ to the whole space X.

The following conditions are satisfied:

I. For every $\tilde{x} \in \widetilde{X}$,

$$\|\varphi A\tilde{x} - \bar{A}\varphi\tilde{x}\| \leq \varepsilon\|\tilde{x}\|.$$

II. For every $x \in X$ one can find $\tilde{x} \in \widetilde{X}$ such that

$$\|Ax - \tilde{x}\| \leq \varepsilon_1\|x\|.$$

In the work of L. V. Kantorovich ([1]) the operators $K = A - \lambda I$ and $\bar{K} = \bar{A} - \lambda I$ are considered, and conditions are given whose fulfillment guarantees the existence of the operator $\bar{K}^{-1}$, if K^{-1} exists, and a set of estimates.

Let a simple eigenvalue λ_0 and an eigenvector x_0 of the operator A be known, as well as an eigenvector f_0 of the adjoint operator A^*. We shall assume that $f_0(x_0) = 1$; consequently, the pair λ_0, x_0 is a solution of the system

$$Ax - \lambda x = 0,$$

$$f_0(x) - 1 = 0.$$

If we introduce the space U, whose elements are pairs

$$u = \begin{pmatrix} x \\ \lambda \end{pmatrix} \quad (x \in X;\ \lambda \text{ is a complex number};\ \|u\|^2 = \|x\|^2 + |\lambda|^2),$$

then this system can be written in the form of a single nonlinear functional equation in the space U:

$$P\begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} Ax - \lambda x \\ f_0(x) - 1 \end{pmatrix} = 0. \tag{1}$$

We normalize the eigenvector of the operator $\bar{A}$ in the following way:

$$f_0(\varphi_0^{-1}\bar{x}) = 1.$$

Then to equation (1) one can put in correspondence the following equation in the space $\bar{U}$ of elements

$$\bar{u} = \begin{pmatrix} \bar{x} \\ \lambda \end{pmatrix} \quad (\bar{x} \in \bar{X};\ \lambda \text{ is a complex number}):$$

$$\bar{P}\begin{pmatrix} \bar{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} \bar{A}\bar{x} - \lambda \bar{x} \\ f_0(\varphi_0^{-1}\bar{x}) - 1 \end{pmatrix} = 0. \tag{2}$$

The space U contains the subspace

$$\widetilde{U} = \left\{ \begin{pmatrix} \tilde{x} \\ \lambda \end{pmatrix} \right\} \quad (\tilde{x} \in \widetilde{X}),$$

isomorphic to $\bar{U}$. The isomorphism is carried out by means of the operation

$$\Phi_0 \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} \varphi x \\ \lambda \end{pmatrix},$$

having the inverse

$$\Phi_0^{-1}\begin{pmatrix} \bar{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} \varphi_0^{-1}\bar{x} \\ \lambda \end{pmatrix}; \qquad \Phi\begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} \varphi x \\ \lambda \end{pmatrix}$$

is an extension of Φ_0 to the whole space U.

Thus, the determination of the eigenvalue $\bar{\lambda}_0$ and eigenvector $\bar{x}_0$ of the operator $\bar{A}$ is reduced to solving a nonlinear equation. In solving it one may use the analogue, proposed by L. V. Kantorovich, of Newton' s method for solving functional equations [1], for which the most essential requirement is the existence of an operation inverse to the derivative operation $\bar{P}'_{\bar{u}_0}$, computed for the initial approximation of Newton' s method. Taking the element

$$\bar{u}_0 = \begin{pmatrix} \varphi x_0 \\ \lambda_0 \end{pmatrix},$$

all the conditions for its applicability are fulfilled.

The derivatives of the operators $\bar{P}_{\varphi x_0,\lambda_0}$ and P'_{x_0,λ_0} have the form:

$$\bar{P}'_{\varphi x_0,\lambda_0}\begin{pmatrix}\bar{x}\\ \lambda\end{pmatrix} = \begin{pmatrix}\bar{A}\bar{x} - \lambda_0\bar{x} - \lambda\varphi x_0\\ f_0\left(\varphi_0^{-1}\bar{x}\right)\end{pmatrix},$$

$$P'_{x_0,\lambda_0}\begin{pmatrix}x\\ \lambda\end{pmatrix} = \begin{pmatrix}Ax - \lambda_0 x - \lambda x_0\\ f_0(x)\end{pmatrix},$$

and it is easy to show that the latter operator has the inverse

$$\left(P'_{x_0,\lambda_0}\right)^{-1}\begin{pmatrix}y\\ \mu\end{pmatrix} = \begin{pmatrix}Ry + \mu x_0\\ -f_0(y)\end{pmatrix}, \qquad \left\|\left(P'_{x_0,\lambda_0}\right)^{-1}\right\|^2 \le \|R\|^2 + \|x_0\|^2 + \|f_0\|^2.$$

Here the following notation has been adopted: R is the resolvent operator $R = (A - \lambda_0 I)^{-1}$, defined and bounded on the set of those x for which $f_0(x) = 0$, and equal to zero on the eigensubspace.

The operators $\bar{P}_{\varphi x_0,\lambda_0}$ and P'_{x_0,λ_0} are connected by conditions analogous to conditions I and II:

I.

$$\begin{aligned}&\left\|\Phi P'_{x_0,\lambda_0}\begin{pmatrix}\tilde{x}\\ \lambda\end{pmatrix} - \bar{P}'_{\varphi x_0,\lambda_0}\Phi\begin{pmatrix}\tilde{x}\\ \lambda\end{pmatrix}\right\|\\ &= \left\|\begin{pmatrix}\varphi A\tilde{x} - \lambda_0\varphi\tilde{x} - \lambda\varphi x_0\\ f_0(\tilde{x})\end{pmatrix} - \begin{pmatrix}\bar{A}\varphi\tilde{x} - \lambda_0\varphi\tilde{x} - \lambda\varphi x_0\\ f_0\left(\varphi_0^{-1}\varphi\tilde{x}\right)\end{pmatrix}\right\|\\ &= \left\|\varphi A\tilde{x} - \bar{A}\varphi\tilde{x}\right\| \le \varepsilon\|\tilde{x}\| \le \varepsilon\left\|\begin{pmatrix}\tilde{x}\\ \lambda\end{pmatrix}\right\|.\end{aligned}$$

II. It is necessary to approximate, by elements of the space $\widetilde{U}$, the elements

$$P'\begin{pmatrix}x\\ \lambda\end{pmatrix} + \lambda_0\begin{pmatrix}x\\ \lambda\end{pmatrix} = \begin{pmatrix}Ax - \lambda x_0\\ f_0(x) + \lambda\lambda_0\end{pmatrix}.$$

Find $\tilde{x}_1$ so that $\|Ax - \tilde{x}_1\| \le \varepsilon_1\|x\|$, and $\tilde{x}_0$ so that $\|Ax_0 - \tilde{x}_0\| \le \varepsilon_1\|x_0\|$. Then for

$$\tilde{x} = \tilde{x}_1 - \frac{\lambda}{\lambda_0}\tilde{x}_0 \quad \text{and} \quad \tilde{\lambda} = -f_0(x) - \lambda\lambda_0$$

we have:

$$\left\| \begin{pmatrix} Ax - \lambda x_0 \\ f_0(x) + \lambda\lambda_0 \end{pmatrix} - \begin{pmatrix} \tilde{x} \\ \tilde{\lambda} \end{pmatrix} \right\| = \|Ax - \lambda x_0 - \tilde{x}\| \le \|Ax - \tilde{x}_1\| + \left\| \lambda x_0 - \frac{\lambda}{\lambda_0}\tilde{x}_0 \right\| \le$$

$$\le \varepsilon_1 \|x\| + \left\| \frac{\lambda}{\lambda_0} A x_0 - \frac{\lambda}{\lambda_0}\tilde{x}_0 \right\| \le \varepsilon_1 \left(1 + \frac{\|x_0\|^2}{|\lambda_0|^2}\right)^{1/2} \left\| \begin{pmatrix} x \\ \lambda \end{pmatrix} \right\|.$$

If the quantity

$$q = \left\{ \varepsilon + \varepsilon_1 \sqrt{1 + \frac{\|x_0\|^2}{|\lambda_0|^2}} \left(\varepsilon + \|\Phi\| \sqrt{\|A - \lambda_0 I\|^2 + \|f_0\|^2 + \|x_0\|^2} \right) \right\} \times$$

$$\times \|\Phi_0^{-1}\| \sqrt{\|R\|^2 + \|f_0\|^2 + \|x_0\|^2} < 1,$$

then, according to the theorems of the general theory of approximate methods ([1]), from the existence of $(P'_{x_0,\lambda_0})^{-1}$ there follows the existence of $(\bar{P}'_{\varphi x_0,\lambda_0})^{-1}$, and

$$\left\| (\bar{P}'_{\varphi x_0,\lambda_0})^{-1} \right\| \le \frac{\left(1 + \sqrt{1 + \frac{\|x_0\|^2}{|\lambda_0|^2}}\right) \|\Phi\| \, \|\Phi_0^{-1}\| \sqrt{\|R\|^2 + \|f_0\|^2 + \|x_0\|^2}}{1 - q} \equiv B_0.$$

Let us estimate $\bar{P}\binom{\varphi x_0}{\lambda_0}$:

$$\bar{P}\begin{pmatrix} \varphi x_0 \\ \lambda_0 \end{pmatrix} = \begin{pmatrix} \bar{A}\varphi x_0 - \lambda_0 \varphi x_0 \\ f_0(\varphi_0^{-1}\varphi x_0) - 1 \end{pmatrix} = \begin{pmatrix} \bar{A}\varphi x_0 - \varphi A x_0 \\ f_0(\varphi_0^{-1}\varphi x_0) - f_0(x_0) \end{pmatrix}.$$

One can find $\tilde{x}_0 \in \tilde{X}$ such that

$$\left\| x_0 - \frac{\tilde{x}_0}{\lambda_0} \right\| = \left\| \frac{A x_0}{\lambda_0} - \frac{\tilde{x}_0}{\lambda_0} \right\| \le \varepsilon_1 \left| \frac{\|x_0\|}{\lambda_0} \right|.$$

Denote $\tilde{x}'_0 = \dfrac{\tilde{x}_0}{\lambda_0}$; $\|\tilde{x}'_0\| \le \left(1 + \dfrac{\varepsilon_1}{|\lambda_0|}\right) \|x_0\|$;

$$\begin{aligned} \|\bar{A}\varphi x_0 - \varphi A x_0\| &= \|\bar{A}\varphi x_0 - \bar{A}\varphi \tilde{x}'_0 - \bar{A}\varphi \tilde{x}'_0 + \varphi A \tilde{x}'_0 - \varphi A \tilde{x}'_0 - \varphi A x_0\| \\ &\le \varepsilon_1 \left| \frac{\|x_0\|}{\lambda_0} \right| (\|\bar{A}\varphi\| + \|\varphi A\|) + \varepsilon \|\tilde{x}'_0\| \\ &\le \left[\varepsilon_1 (\|\bar{A}\varphi\| + \|\varphi A\| + \varepsilon) + \varepsilon |\lambda_0| \right] \left| \frac{\|x_0\|}{\lambda_0} \right|. \end{aligned}$$

Further,

$$|\,\varphi_0^{-1}\varphi x_0 - x_0\,| \le \|\varphi_0^{-1}\varphi x_0 - \varphi_0^{-1}\varphi\tilde{x}_0'\| + \|\tilde{x}_0' - x_0\| \le (\|\varphi_0^{-1}\varphi\| + 1)\|\tilde{x}_0' - x_0\| \le \varepsilon_1(1 + \|\varphi_0^{-1}\varphi\|)\left|\frac{\|x_0\|}{\lambda_0}\right|.$$

Hence

$$\left\|\bar{P}\begin{pmatrix}\varphi x_0\\ \lambda_0\end{pmatrix}\right\|^2 \le \left|\frac{\|x_0\|^2}{\lambda_0}\right|\left\{\left[\varepsilon_1(\varepsilon + \|\bar{A}\varphi\| + \|\varphi A\|) + \varepsilon|\lambda_0|\right]^2 + \varepsilon_1^2(1 + \|\varphi_0^{-1}\varphi\|)^2\|f_0\|^2\right\} \equiv \eta_0^2.$$

The second derivative P_u'' is bounded everywhere,

$$\|P_u''\| \le \sqrt{3/2}.$$

It remains necessary that the condition $h_0 = \sqrt{3/2}\,B_0^2\eta_0 \le 1/2$ be fulfilled, and then application of the theorems of Newton' s method ([1]) gives the following result:

Theorem 1. *Let λ_0 be a simple eigenvalue of the operator A; x_0 its eigenvector; f_0 an eigenvector of the adjoint operator, with $f_0(x_0) = 1$.*

If conditions I and II are satisfied,

$$q = \left\{\varepsilon + \varepsilon_1\sqrt{1 + \frac{\|x_0\|^2}{|\lambda_0|^2}}\left(\varepsilon + \|\Phi\|\sqrt{\|A - \lambda_0 I\|^2 + \|f_0\|^2 + \|x_0\|^2}\right)\right\}\cdot\|\Phi_0^{-1}\|\sqrt{\|R\|^2 + \|f_0\|^2 + \|x_0\|^2} < 1;$$

$$h_0 = \sqrt{\frac{3}{2}}\,\frac{1}{|\lambda_0|}\,\frac{\left(1 + \varepsilon_1\sqrt{1 + \frac{\|x_0\|^2}{|\lambda_0|^2}}\right)^2\|\Phi\|^2\|\Phi_0^{-1}\|^2(\|R\|^2 + \|f_0\|^2 + \|x_0\|^2)}{(1-q)^2}$$
$$\times\left\{\left[\varepsilon|\lambda_0| + \varepsilon_1(\varepsilon + \|\bar{A}\varphi\| + \|\varphi A\|)\right]^2 + \varepsilon_1^2(1 + \|\varphi_1^{-1}\varphi\|)^2\|f_0\|^2\right\}^{1/2} \le \frac{1}{2},$$

then in the domain

$$\left\|\begin{pmatrix}\bar{x}\\ \lambda\end{pmatrix} - \begin{pmatrix}\varphi x_0\\ \lambda_0\end{pmatrix}\right\| \le \frac{1 - \sqrt{1 - 2h_0}}{h_0}\,\eta_0$$

the operator $\bar{A}$ has a unique eigenvalue $\bar{\lambda}_0$ and a corresponding unique eigenvector $\bar{x}_0$ such that $f_0(\varphi_0^{-1}x_0) = 1$.

The closeness of the solutions can be estimated in the form

$$\left\|\begin{pmatrix}\varphi_0^{-1}\bar{x}_0\\ \bar{\lambda}_0\end{pmatrix}-\begin{pmatrix}x_0\\ \lambda_0\end{pmatrix}\right\|\le\frac{1-\sqrt{1-2h_0}}{h_0}\,\eta_0\|\Phi_0^{-1}\|+\varepsilon_1(1+\|\Phi_0^{-1}\|\|\Phi\|)\left|\frac{\|x_0\|}{\lambda_0}\right|.$$

Quite analogously, the second result is obtained:

Theorem 2. Let $\overline{\lambda}_0$ be a simple eigenvalue of the operator $\overline{A}$; $\overline{x}_0$ its eigen-element; f_0 an eigen-element of the adjoint operator. If conditions I and II are satisfied,

$$r=\frac{\varepsilon_1}{|\overline{\lambda}_0|}\left(1+\|\Phi_0^{-1}\|\,\|\Phi P'_{\varphi_0^{-1}x_0,\lambda_0}\|\sqrt{\|\overline{R}\|^2+\|\overline{f}_0\|^2+\|\overline{x}_0\|^2}+2\varepsilon\|\Phi_0^{-1}\|\sqrt{\|\overline{R}\|^2+\|\overline{f}_0\|^2+\|\overline{x}_0\|^2}\right)<1,$$

$$h_0=\sqrt{\frac{3}{2}\,\frac{\left[\|\Phi_0^{-1}(\overline{P}^1_{\overline{x}0,\lambda_0})^{-1}\Phi\|+\frac{2}{|\overline{\lambda}_0|}\left(1+\|\Phi_0^{-1}\|\,\|\Phi P'_{\varphi_0^{-1}x_0,\lambda_0}\|\sqrt{\|\overline{R}\|^2+\|\overline{x}_0\|^2+\|\overline{f}_0\|^2}\right)\right]^2}{(1-r)^2}}\times[\varepsilon_1+(\varepsilon+\varepsilon_1\|\varphi\|)\|\varphi_0^{-1}$$

then the operator A has, in the domain

$$\left\|\begin{pmatrix}x\\ \lambda\end{pmatrix}-\begin{pmatrix}\varphi_0^{-1}\overline{x}_0\\ \overline{\lambda}_0\end{pmatrix}\right\|\le\frac{1-\sqrt{1-2h_0}}{h_0}\,[\varepsilon_1+(\varepsilon+\varepsilon_1\|\varphi\|)\|\varphi_0^{-1}\|\,\|\varphi_0^{-1}\|\,\|\overline{x}_0\|]$$

a unique eigen-pair (x_0,λ_0).

Leningrad Branch of the Mathematical Institute
named after V. A. Steklov
Academy of Sciences of the USSR

Received
1 XI 1956

REFERENCES CITED

1. L. V. Kantorovich, *Uspekhi Mat. Nauk*, **3**, no. 6 (28) (1948).

Note: Figure translations are in progress. See original paper for figures.