ON EQUATIONS OF THE TYPE OF THE EQUATIONS OF NONSTATIONARY FILTRATION

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Abstract

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ON EQUATIONS OF THE TYPE OF THE EQUATIONS OF NONSTATIONARY FILTRATION

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The equations of nonstationary filtration of liquids and gases, the equations of heat propagation with thermal conductivity depending on temperature, and boundary-layer equations have been studied by many authors ([1−5]). These equations are nonlinear equations of parabolic type, degenerating for certain values of the sought solution. The Cauchy problem and boundary-value problems for such equations with initial and boundary conditions under which the equation degenerates have been studied only in special cases ([1−5]). If, for the prescribed initial and boundary conditions, the equation does not degenerate, then these problems are a special case of the problems considered in ([6]).

We shall consider the Cauchy problem for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \varphi(t,x,u)}{\partial x^2} \tag{1}$$

with the initial condition

$$u\big|_{t=0} = u_0(x), \tag{2}$$

where $u_0(x) \geq 0$ and is bounded for all x; $\varphi_u \geq 0$ for $u \geq 0$ and $\varphi_u > 0$ for $u > 0$. If $\varphi \equiv u^n$, equation (1) is the equation of nonstationary filtration. The methods that we use to solve the Cauchy problem are also applicable to the study of boundary-value problems and of equations of this type with any number of independent variables, in particular equations of nonstationary filtration in an inhomogeneous medium,

$$\frac{\partial u}{\partial t} = \Delta\varphi(t,x,y,z,u). \tag{3}$$

We shall first give the definition of a generalized solution of the Cauchy problem for equation (1) with condition (2), prove the uniqueness and existence of such a solution, and then show that the generalized solution $u(t,x)$ has the continuous derivatives entering equation (1) and satisfies this equation at all points where $u(t,x) \neq 0$. We shall assume that $\varphi(t,x,u)$ is a continuous function of its arguments and that φ_u is bounded for $0 \leq t \leq T$, $-\infty < x < +\infty$, and bounded u.

A function $u(t,x)$, continuous and bounded in the domain $G\{0 \leq t \leq T,\ -\infty < x < +\infty\}$, will be called a **generalized solution of the Cauchy problem for equation** (1) **with condition** (2), if there exists a generalized derivative $\partial\varphi(t,x,u(t,x))/\partial x$, bounded in G, and if, for every continuously differentiable function $f(t,x)$ in G, equal to zero outside a finite domain lying in the half-plane $t \leq T$, the equality

$$\iint_G \left[\frac{\partial f}{\partial t}\, u(t,x) - \frac{\partial f}{\partial x}\frac{\partial\varphi(t,x,u(t,x))}{\partial x}\right] dx\, dt + \int_{-\infty}^{+\infty} f(0,x)u_0(x)\, dx = 0, \quad (4)$$

holds.

Proof of the uniqueness of the generalized solution of the Cauchy problem. It is easy to show that equality (4) is also satisfied for any function $f(t,x)$ equal to zero at $t = T$ and outside a finite domain, continuous in G, and having in G bounded generalized derivatives $\partial f/\partial t$ and $\partial f/\partial x$. This follows from the fact that for the mean functions $\bar{f}_h$ of the function f ([7], pp. 19-20, 39-48) this equality is satisfied by the definition of a generalized solution, and in it one may pass to the limit as $h \to 0$. Let $u_1(t,x)$ and $u_2(t,x)$ be two generalized solutions of problem (1), (2). For any function f of the class indicated above, the relation

$$\iint_G \left\{\frac{\partial f}{\partial t}(u_1 - u_2) - \frac{\partial f}{\partial x}\left[\frac{\partial\varphi(t,x,u_1)}{\partial x} - \frac{\partial\varphi(t,x,u_2)}{\partial x}\right]\right\} dx\, dt = 0. \quad (5)$$

must hold.

To show that $u_1 \equiv u_2$, take in relation (5)

$$f(t,x) = -\alpha_n(x)\int_T^t [\varphi(\tau,x,u_1(\tau,x)) - \varphi(\tau,x,u_2(\tau,x))]\, d\tau,$$

where $\alpha_n(x) = 1$ for $|x| \leq n-1$, and $\alpha_n(x) = 0$ for $|x| \geq n$; $0 \leq \alpha_n(x) \leq 1$; $\alpha_n'(x)$ are uniformly bounded with respect to n. Clearly, the function f so constructed is continuous in G and has bounded generalized derivatives with respect to t and x. For such a function f, from (5) we obtain

$$\iint_G \alpha_n(x)[\varphi(t,x,u_1) - \varphi(t,x,u_2)](u_1 - u_2)\, dx\, dt -$$

$$-\frac{1}{2}\iint_G \alpha_n(x)\frac{d}{dt}\left(\int_T^t \left[\frac{\partial\varphi(\tau,x,u_1)}{\partial x}-\frac{\partial\varphi(\tau,x,u_2)}{\partial x}\right]d\tau\right)^2 dx\,dt-$$

$$-\iint_G \left\{\alpha_n'(x)\left[\int_T^t [\varphi(\tau,x,u_1)-\varphi(\tau,x,u_2)]\,d\tau\right]\left[\frac{\partial\varphi(t,x,u_1)}{\partial x}-\frac{\partial\varphi(t,x,u_2)}{\partial x}\right]\right\}dx\,dt=0. \tag{6}$$

Let Q_n be the rectangles $\{\, n-1\le |x|\le n,\ 0\le t\le T\,\}$. From equality (6) it follows that

$$\iint_G \alpha_n(x)[\varphi(t,x,u_1)-\varphi(t,x,u_2)](u_1-u_2)\,dx\,dt+$$

$$+\frac{1}{2}\int_{-\infty}^{+\infty}\alpha_n(x)\left[\int_T^0\left(\frac{\partial\varphi(\tau,x,u_1)}{\partial x}-\frac{\partial\varphi(\tau,x,u_2)}{\partial x}\right)d\tau\right]^2 dx=$$

$$=\iint_{Q_n}\left\{\alpha_n'(x)\left[\int_T^t [\varphi(\tau,x,u_1)-\varphi(\tau,x,u_2)]\,d\tau\right]\times\right.$$

$$\left.\times\left(\frac{\partial\varphi(t,x,u_1)}{\partial x}-\frac{\partial\varphi(t,x,u_2)}{\partial x}\right)\right\}dx\,dt. \tag{7}$$

It is easy to see that the integral appearing on the right-hand side of (7) is bounded uniformly with respect to n. Since the integrands in the integrals on the left-hand side of (7) are nonnegative, each of them tends to some finite limit as $n\to\infty$. This means that the function

$$\Phi(t,x)=(u_1-u_2)[\varphi(t,x,u_1)-\varphi(t,x,u_2)]$$

is summable in G. We shall show that the right-hand side of (7) tends to zero as $n\to\infty$. Indeed, by virtue of our assumptions, for this integral I we have

$$|I|\le C_1T\iint_{Q_n}|\varphi(t,x,u_1)-\varphi(t,x,u_2)|\,dx\,dt\le$$

$$\le 2C_1T^{3/2}\left(\iint_{Q_n}|\varphi(t,x,u_1)-\varphi(t,x,u_2)|^2\,dx\,dt\right)^{1/2}=$$

$$= 2C_1 T^{3/2} \left(\iint_{Q_n} (u_1 - u_2)[\varphi(t,x,u_1) - \varphi(t,x,u_2)]\varphi'_u(t,x,\tilde{u})\, dx\, dt \right)^{1/2} \le$$

$$\le C_2 \left(\iint_{Q_n} \Phi\, dx\, dt \right)^{1/2}.$$

The last integral tends to zero as $n \to \infty$, by virtue of the summability of Φ in G. Hence it follows that the integrals on the left-hand side of (7) are equal to zero. From the equality $\Phi \equiv 0$ it follows that $u_1 = u_2$.

Remarks. 1. A generalized solution $u(t,x,y,z)$ of the Cauchy problem for equation (3) with the condition $u|_{t=0} = u_0(x,y,z)$ may be defined analogously. The function $u(t,x,y,z)$ is required to be continuous, to have bounded generalized derivatives with respect to x, y, z of the function $\varphi(t,x,y,z,u(t,x,y,z))$, and to satisfy an equality analogous to (4).

The uniqueness proof given above carries over completely to the case of any number of spatial variables.

2. Consider the boundary-value problem for equation (1) with the conditions

$$u(0,x) = u_0(x) \ge 0, \qquad u(t,0) = u_1(t) \ge 0 \tag{8}$$

in the domain $G_1\{0 \le t \le T,\ 0 \le x < +\infty\}$.

A function $u(t,x)$, continuous and bounded in the domain G_1, satisfying conditions (8), will be called a **generalized solution** of problem (1), (8), if there exists a bounded generalized derivative $\partial\varphi(t,x,u(t,x))/\partial x$ in G_1, and equality (4) is fulfilled for $G = G_1$ and any continuously differentiable in G_1 function $f(t,x)$ equal to zero for $t = T$, for $x \le 0$, and at all points outside some finite domain.

The proof of uniqueness of the solution of the Cauchy problem carries over to the case of problem (1), (8).

Proof of existence of a solution of the Cauchy problem. We shall assume that $\varphi = \varphi(x,u)$; $u_0(x)$ is a continuous function, $|u_0(x)| \le M$; $\varphi(x,0) = 0$; $\varphi(x,u_0(x))$ satisfies the Lipschitz condition for $-\infty < x < +\infty$; $\varphi(x,u)$ has continuous derivatives of fifth order with respect to x and u in $R\{-\infty < x < +\infty,\ 0 \le u \le M + \varepsilon = M_1\}$, where $\varepsilon > 0$, and $\varphi(x,u)$ is bounded in R. Consider the function $v_0(x) = \varphi(x,u_0(x))$ and construct a monotonically decreasing sequence of four times continuously differentiable functions $\{v_0^n(x)\}$, converging uniformly to $v_0(x)$ and such that $|dv_0^n/dx| \le K$ and $0 < v_0^n(x) \le M_2$ for all n, where $M_2 = \max\varphi(x,M_1)$, K is some constant. Construct a sequence $\{\bar{v}_n(x)\}$

as follows: $\bar{v}_n(x) = v_0^n(x)$ for $|x| \le n-2$; $\bar{v}_n(x) = M_2$ for $|x| \ge n-1$, and $v_0^n(x) \le \bar{v}_n(x) \le M_2$, $|d\bar{v}_n/dx| \le K$ and $\bar{v}_n \ge \bar{v}_{n+1}$ for all x and n.

Consider, for the equation

$$\frac{\partial \Phi(x,v)}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \tag{9}$$

where $v \equiv \varphi(x, \Phi(x,v))$, the first boundary-value problem in the rectangle $G_n\{0 \le t \le T,\ -n < x < n\}$ with the conditions

$$v(0,x) = \bar{v}_n(x), \qquad v(t,n) = M_2, \qquad v(t,-n) = M_2.$$

The existence of a solution $v_n(t,x)$ of this problem under the conditions that are satisfied under our assumptions was proved in ([6]), since $\Phi'_v > 0$ for $v > 0$. For equation (9) the maximum principle is valid. Therefore $0 < v_n(t,x) \le M_2$. Since on the boundary $\Gamma\{t=0,\ x=\pm n\}$ of the domain G_n one has $v_{n+1} \le v_n$, it follows easily from the equation satisfied by the difference $v_{n+1} - v_n$ that $v_{n+1}(t,x) \le v_n(t,x)$ in G_n. For the equation satisfied by $\partial v_n/\partial x$, the maximum principle is valid. Therefore $|\partial v_n/\partial x|$ in G_n does not exceed $\max|\partial v_n/\partial x|$ on Γ_n. The derivatives $\partial v_n/\partial x$ on Γ_n are bounded by a constant independent of n. This is easily proved analogously to the way $\partial u_n/\partial x$ is estimated for $x=0$ and $x=1$ in ([6]).

We now pass from equation (9) to (1). The functions $u_n = \Phi(x, v_n)$ form a monotonically decreasing sequence, satisfy equation (1), and $|\partial\varphi(x,u_n)/\partial x| \le K_2$ for all n. Let

$$u(t,x) = \lim_{n\to\infty} u_n(t,x).$$

We shall show that $u(t,x)$ is a generalized solution of problem (1), (2). The existence in G of a bounded generalized derivative $\partial\varphi(x,u(t,x))/\partial x$ follows from the condition

$$\left|\frac{\partial\varphi(x,u_n(t,x))}{\partial x}\right| \le K_2.$$

Equality (4) is satisfied for the function $u(t,x)$, since it is satisfied for $u_n(t,x)$, and in it one may pass to the limit as $n \to \infty$ by virtue of the existence of a subsequence $\partial\varphi(x,u_n(t,x))/\partial x$ weakly converging in every finite domain, as $n \to \infty$, to $\partial\varphi(x,u(t,x))/\partial x$. To prove the continuity of $u(t,x)$ in G, it suffices to prove the continuity of $\varphi(x,u(t,x))$ with respect to t, since $\varphi(x,u(t,x))$ satisfies a Lipschitz condition in x with a constant independent of t. If $\varphi(x,u(t,x))$ were discontinuous at a point (t_0,x_0), then there would exist a sequence $t_n \to t_0$ such that, for $|x_0 - x| \le \delta$,

$$|u(t_n,x) - u(t_0,x)| > \varepsilon \qquad (\delta > 0,\ \varepsilon > 0).$$

Consider a smooth nonnegative function $f(t,x)$, equal to zero outside the domain $|x_0 - x| \le \delta,\ |t_0 - t| \le \delta$. For the functions f and $u(t,x)$, in any strip $\{t_n \le$

$t \leq T,\ -\infty < x < +\infty\}$ there holds a relation analogous to (4). Subtracting from one another the equalities (4) for the strip $\{t_n \leq t \leq T,\ -\infty < x < +\infty\}$ and the strip

$$\{t_0 \leq t \leq T,\ -\infty < x < +\infty\},$$

we easily obtain that

$$\int_{-\infty}^{+\infty} f(t_0, x)\,[u(t_0, x) - u(t_n, x)]\,dx \to 0$$

as $n \to \infty$, which is impossible. Hence it also follows that $u(t, x)$ satisfies condition (2).

The existence of continuous derivatives of first and second order of the function $u(t, x)$ at points where $u(t, x) \neq 0$ follows from the fact that in a neighborhood of each such point $v_n(t, x) \geq \alpha > 0$, and the uniform boundedness with respect to n of the derivatives of $v_n(t, x)$ with respect to x up to the fourth order in this neighborhood can be obtained by using the well-known method of S. N. Bernstein ([8]).

Analogously one can prove the existence of a solution of problem (1), (8) for $u_1(t) = 0$.

As the examples of solutions constructed in ([3]) and the uniqueness theorem proved above show, problem (1), (2) and problem (1), (8) may fail to have a smooth solution for all (t, x) in G.

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Note: Figure translations are in progress. See original paper for figures.